# Chapter 3

# Graphing and Maximum–Minimum Problems

*Differential calculus provides tests for locating the key features of graphs.*

Now that we know how to differentiate, we can use this information to assist us in plotting graphs. The signs of the derivative and the second derivative of a function will tell us which way the graph of the function is "leaning" and "bending."

Using the derivative to predict the behavior of graphs helps us to find the points where a function takes on its maximum and minimum values. Many interesting word problems requiring the "best" choice of some variable involve searching for such points.

In Section 3.1, we study the geometric aspects of continuity. This will provide a useful introduction to graphing. In Section 3.6, we use the ideas of maxima and minima to derive an important theoretical result—the mean value theorem. One consequence of this theorem is a fact which we used in connection with antiderivatives: a function whose derivative is zero must be constant.

## 3.1 Continuity and the Intermediate Value Theorem

*If a continuous function on a closed interval has opposite signs at the endpoints, it must be zero at some interior point.*

In Section 1.2, we defined continuity as follows: "A function $f(x)$ is said to be continuous at $x_0$ if $\lim_{x \to x_0} f(x) = f(x_0)$." A function is said to be *continuous on a given interval* if it is continuous at every point on that interval. If a function $f$ is continuous on the whole real line, we just say that "$f$ is continuous." An imprecise but useful guide is that a function is continuous when its graph can be drawn "without removing pencil from paper." In Figure 3.1.1, the curve on the left is continuous at $x_0$ while that on the right is not.

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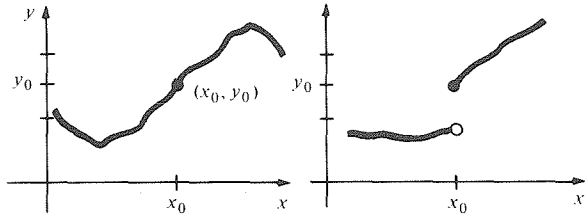


**Figure 3.1.1.** Illustrating a continuous curve (left) and a discontinuous curve (right).

**Example 1** Decide where each of the functions whose graphs appear in Fig. 3.1.2 is continuous. Explain your answers.

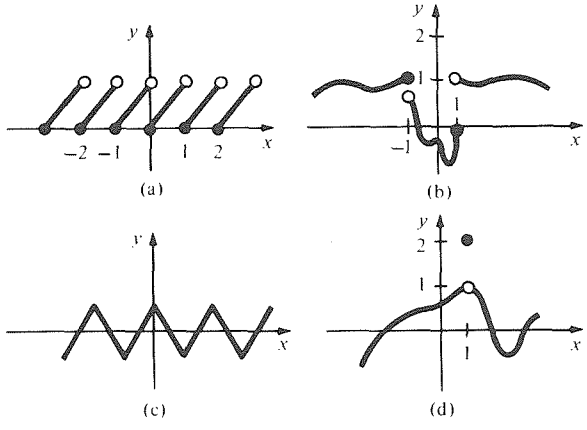


**Figure 3.1.2.** Where are these functions continuous?

**Solution** (a) This function jumps in value at each of the points $x_0 = 0$, $x_0 = \pm 1$, $x_0 = \pm 2, \ldots,$ so $\lim_{x \to x_0} f(x)$ does not exist at these points and thus $f$ is not continuous there; however, $f$ is continuous on each of the intervals between the jump points.

(b) This function jumps in value at $x_0 = -1$ and $x_0 = +1$, and so $\lim_{x \to +1} f(x)$ and $\lim_{x \to -1} f(x)$ do not exist. Thus $f$ is not continuous at $x_0 = \pm 1$; it is continuous on each of the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

(c) Even though this function has sharp corners on its graph, it *is* continuous; $\lim_{x \to x_0} f(x) = f(x_0)$ at each point $x_0$.

(d) Here $\lim_{x \to 1} f(x) = 1$, so the limit exists. However, the limit does not equal $f(1) = 2$. Thus $f$ is *not* continuous at $x_0 = 1$. It is continuous on the intervals $(-\infty, 1)$ *and* $(1, \infty)$. ▲

In Section 1.2, we used various limit theorems to establish the continuity of functions that are basic to calculus. For example, the rational function rule for limits says that a *rational function is continuous at points where its denominator does not vanish.*

**Example 2** Show that the function $f(x) = (x - 1)/3x^2$ is continuous at $x_0 = 4$.

**Solution** This is a rational function whose denominator does not vanish at $x_0 = 4$, so it is continuous by the rational function rule. ▲

**Example 3** Let $g(x)$ be the *step function* defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Show that $g$ is not continuous at $x_0 = 0$. Sketch.

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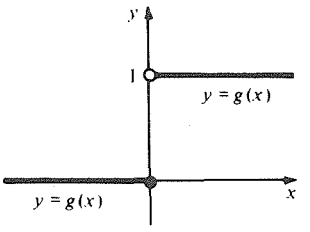


**Figure 3.1.3.** This step function is discontinuous at $x_0 = 0$.

**Solution** The graph of $g$ is shown in Fig. 3.1.3. Since $g$ approaches (in fact, equals) 0 as $x$ approaches 0 from the left, but approaches 1 as $x$ approaches 0 from the right, $\lim_{x\to 0} g(x)$ does not exist. Therefore, $g$ is not continuous at $x_0 = 0$. ▲

**Example 4** Using the laws of limits, show that if $f$ and $g$ are continuous at $x_0$, so is $fg$.

**Solution** We must show that $\lim_{x\to x_0}(fg)(x) = (fg)(x_0)$. By the product rule for limits, $\lim_{x\to x_0}[f(x)g(x)] = [\lim_{x\to x_0} f(x)][\lim_{x\to x_0} g(x)] = f(x_0)g(x_0)$, since $f$ and $g$ are continuous at $x_0$. But $f(x_0)g(x_0) = (fg)(x_0)$, and so $\lim_{x\to x_0}(fg)(x) = (fg)(x_0)$, as required. ▲

In Section 1.3, we proved the following theorem: *if $f$ is differentiable at $x_0$, then $f$ is continuous at $x_0$*. Using our knowledge of differential calculus, we can use this relationship to establish the continuity of additional functions or to confirm the continuity of functions originally determined using the laws of limits.

**Example 5** (a) Show that $f(x) = 3x^2/(x^3 - 2)$ is continuous at $x_0 = 1$. Where else is it continuous?

(b) Show that $f(x) = \sqrt{x^2 + 2x + 1}$ is continuous at $x = 0$.

**Solution** (a) By our rules for differentiation, we see that this function is differentiable at $x_0 = 1$; indeed, $x^3 - 2$ does not vanish at $x_0 = 1$. Thus $f$ is also continuous at $x_0 = 1$. Similarly, $f$ is continuous at each $x_0$ such that $x_0^3 - 2 \neq 0$, i.e., at each $x_0 \neq \sqrt[3]{2}$.

(b) This function is the composition of the square root function $h(u) = \sqrt{u}$ and the function $g(x) = x^2 + 2x + 1$; $f(x) = h(g(x))$. Note that $g(0) = 1 > 0$. Since $g$ is differentiable at any $x$ (being a polynomial), and $h$ is differentiable at $u = 1$, $f$ is differentiable at $x = 0$ by the chain rule. Thus $f$ is continuous at $x = 0$. ▲

According to our previous discussion, a continuous function is one whose graph never "jumps." The definition of continuity is *local* since continuity at each point involves values of the function only near that point. There is a corresponding *global* statement, called the intermediate value theorem, which involves the behavior of a function over an entire interval $[a, b]$.

## Intermediate Value Theorem (First Version)

Let $f$ be continuous[1] on $[a, b]$ and suppose that, for some number $c$, $f(a) < c < f(b)$ or $f(a) > c > f(b)$. Then there is some point $x_0$ in $(a, b)$ such that $f(x_0) = c$.

[1] Our definition of continuity on $[a, b]$ assumes that $f$ is defined near each point $\bar{x}$ of $[a, b]$, including the endpoints, and that $\lim_{x\to\bar{x}} f(x) = f(\bar{x})$. Actually, at the endpoints, it is enough to assume that the one-sided limits (from inside the interval) exist, rather than the two-sided ones.

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In geometric terms, this theorem says that for the graph of a continuous function to pass from one side of a horizontal line to the other, the graph must meet the line somewhere (see Fig. 3.1.4). The proof of the theorem depends on a careful study of properties of the real numbers and will be omitted. (See the references listed in the Preface.) However, by drawing additional graphs like those in Fig. 3.1.4, you should convince yourself that the theorem is reasonable.

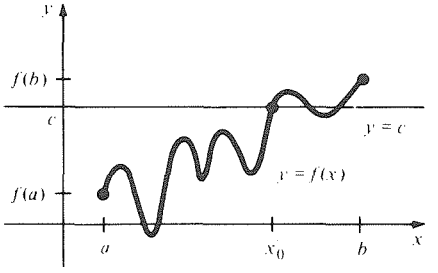


**Figure 3.1.4.** The graph of $f$ must pierce the horizontal line $y = c$ if it is to get across.

**Example 6** Show that there is a number $x_0$ such that $x_0^5 - x_0 = 3$.

**Solution** Let $f(x) = x^5 - x$. Then $f(0) = 0$ and $f(2) = 30$. Since $0 < 3 < 30$, the intermediate value theorem guarantees that there is a number $x_0$ in $(0, 2)$ such that $f(x_0) = 3$. (The function $f$ is continuous on $[0, 2]$ because it is a polynomial.) ▲

Notice that the intermediate value theorem does not tell us *how* to find the number $x_0$ but merely that it *exists*. (A look at Fig. 3.1.4 should convince you that there may be more than one possible choice for $x_0$.) Nevertheless, by repeatedly dividing an interval into two or more parts and evaluating $f(x)$ at the dividing points, we can solve the equation $f(x_0) = c$ as accurately as we wish. This *method of bisection* is illustrated in the next example.

**Example 7** **(The method of bisection)** Find a solution of the equation $x^5 - x = 3$ in $(0, 2)$ to within an accuracy of 0.1 by repeatedly dividing intervals in half and testing each half for a root.

**Solution** In Example 6 we saw that the equation has a solution in the interval $(0, 2)$. To locate the solution more precisely, we evaluate $f(1) = 1^5 - 1 = 0$. Thus $f(1) < 3 < f(2)$, so there is a root in $(1, 2)$. Now we bisect $[1, 2]$ into $[1, 1.5]$ and $[1.5, 2]$ and repeat: $f(1.5) \approx 6.09 > 3$, so there is a root in $(1, 1.5)$; $f(1.25) = 1.80 < 3$, so there is a root in $(1.25, 1.5)$; $f(1.375) \approx 3.54 > 3$, so there is a root in $(1.25, 1.375)$; thus $x_0 = 1.3$ is within 0.1 of a root. Further accuracy can be obtained by means of further bisections. (Related techniques for root finding are suggested in the exercises for this section. Other methods are presented in Section 11.4.) ▲

There is another useful way of stating the intermediate value theorem (the *contrapositive* statement).

### Intermediate Value Theorem (Second Version)

Let $f$ be a function which is continuous on $[a, b]$ and suppose that $f(x) \neq c$ for all $x$ in $[a, b]$. If $f(a) < c$, then $f(b) < c$ as well. (See Fig. 3.1.5.) Similarly, if $f(a) > c$, then $f(b) > c$ as well.

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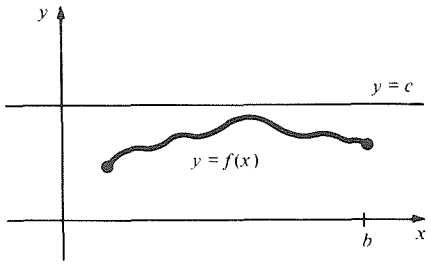


**Figure 3.1.5.** The graph starts below the line $y = c$ and never pierces the line, so it stays below the line.

In geometric language, this second version of the theorem says: "The graph of a continuous function which never meets a horizontal line must remain on one side of it." The first version says: "If the graph of a continuous function passes from one side of a horizontal line to the other, the graph must meet the line somewhere." You should convince yourself that these two statements really mean the same thing.

In practice, the second version of the intermediate value theorem can be useful for determining the sign of a function on intervals where it has no roots, as in the following example.

**Example 8** Suppose that $f$ is continuous on $[0, 3]$, that $f$ has no roots on the interval, and that $f(0) = 1$. Prove that $f(x) > 0$ for *all* $x$ in $[0, 3]$.

**Solution** Apply the intermediate value theorem (version 2), with $c = 0$, to $f$ on $[0, b]$ for each $b$ in $(0, 3]$. Since $f$ is continuous on $[0, 3]$, it is continuous on $[0, b]$; since $f(0) = 1 > 0$, we have $f(b) > 0$. But $b$ was anything in $(0, 3]$, so we have proved what was asked. ▲

## Exercises for Section 3.1

1. Decide where each of the functions whose graph is sketched in Fig. 3.1.6 is continuous.

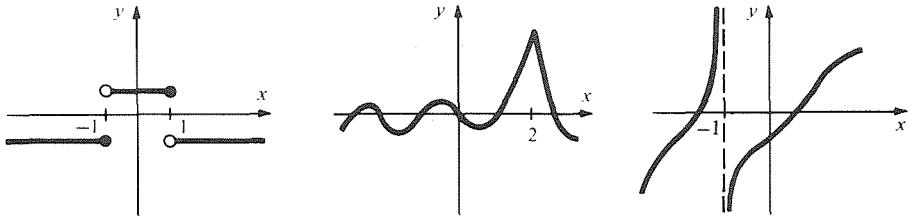


**Figure 3.1.6.** Where are these continuous?

2. Which of the functions in Fig. 3.1.7 are continuous at $x_0 = 1$?

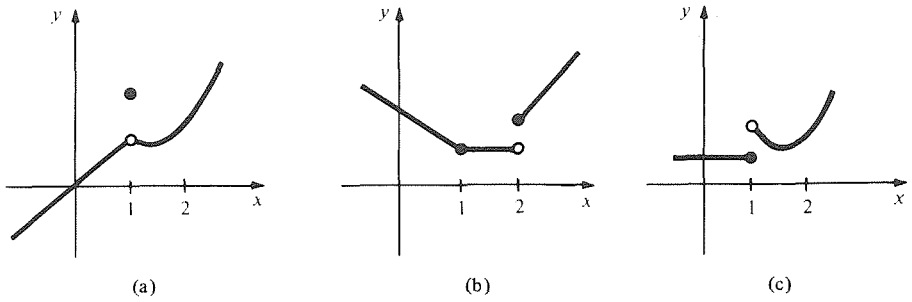


**Figure 3.1.7.** Which of these functions are continuous at $x_0 = 1$?

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3. Show that $f(x) = (x^2 + 1)(x^2 - 1)$ is continuous at $x_0 = 0$.
4. Show that $f(x) = x^3 + 3x^2 - 2x$ is continuous at $x_0 = 3$.
5. Prove that $(x^2 - 1)/(x^3 + 3x)$ is continuous at $x_0 = 1$.
6. Prove that $(x^4 - 8)/(x^3 + 2)$ is continuous at $x_0 = 0$.
7. Where is $(x^2 - 1)/(x^4 + x^2 + 1)$ continuous?
8. Where is $(x^4 + 1)/(x^3 - 8)$ continuous?
9. Let $f(x) = (x^3 + 2)/(x^2 - 1)$. Show that $f$ is continuous on $[-\frac{1}{2}, \frac{1}{2}]$.
10. Is the function $(x^3 - 1)/(x^2 - 1)$ continuous at 1? Explain your answer.
11. Let $f(x)$ be the step function defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ -2 & \text{if } x \geqslant 0. \end{cases}$$

Show that $f$ is discontinuous at 0.

12. Let $f(x)$ be the *absolute value function*: $f(x) = |x|$; that is,

$$f(x) = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

Show that $f$ is continuous at $x_0 = 0$.

13. Using the laws of limits, show that if $f$ and $g$ are continuous at $x_0$, so is $f + g$.
14. If $f$ and $g$ are continuous at $x_0$ and $g(x_0) \neq 0$, show that $f/g$ is continuous at $x_0$.
15. Where is $f(x) = 8x^3/\sqrt{x^2 - 8}$ continuous?
16. Where is $f(x) = 9x^2 - 3x/\sqrt{x^4 - 2x^2 - 8}$ continuous?
17. Show that the equation $-s^5 + s^2 = 2s - 6$ has a solution.
18. Prove that the equation $x^3 + 2x - 1 = 7$ has a solution.
19. Prove that $f(x) = x^8 + 3x^4 - 1$ has at least two distinct zeros.
20. Show that $x^4 - 5x^2 + 1$ has at least two distinct zeros.
21. The roots of $f(x) = x^3 - 2x - x^2 + 2$ are $\sqrt{2}$, $-\sqrt{2}$, and 1. By evaluating $f(-3)$, $f(0)$, $f(1.3)$, and $f(2)$, determine the sign of $f(x)$ on each of the intervals between its roots.
22. Use the method of bisection to approximate $\sqrt{7}$ to within two decimal places. [*Hint:* Let $f(x) = x^2 - 7$. What should you use for $a$ and $b$?]
23. Find a solution of the equation $x^5 - x = 3$ to an accuracy of 0.01.
24. Find a solution of the equation $x^5 - x = 5$ to an accuracy of 0.1.
25. Suppose that $f$ is continuous on $[-1, 1]$ and that $f(x) - 2$ is never zero on $[-1, 1]$. If $f(0) = 0$, show that $f(x) < 2$ for all $x$ in $[-1, 1]$.
26. Suppose that $f$ is continuous on $[3, 5]$ and that $f(x) \neq 4$ for all $x$ in $[3, 5]$. If $f(3) = 3$, show that $f(5) < 4$.
27. Let $f(x)$ be 1 if a certain sample of lead is in the solid state at temperature $x$; let $f(x)$ be 0 if it is in the liquid state. Define $x_0$ to be the melting point of this lead sample. Is there any way to define $f(x_0)$ so as to make $f$ continuous? Give reasons for your answer, and supply a graph for $f$.
28. An empty bucket with a capacity of 10 liters is placed beneath a faucet. At time $t = 0$, the faucet is turned on, and water flows from the faucet at the rate of 5 liters per minute. Let $V(t)$ be the volume of the water in the bucket at time $t$. Present a plausible argument showing that $V$ is a continuous function on $(-\infty, \infty)$. Sketch a graph of $V$. Is $V$ differentiable on $(-\infty, \infty)$?
29. Let $f(x) = (x^2 - 4)/(x - 2)$, $x \neq 2$. Define $f(2)$ so that the resulting function is continuous at $x = 2$.
30. Let $f(x) = (x^3 - 1)/(x - 1)$, $x \neq 1$. How should $f(1)$ be defined in order that $f$ be continuous at each point?
31. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1, \\ ? & \text{if } 1 \leqslant x \leqslant 3, \\ x - 6 & \text{if } 3 < x. \end{cases}$$

How can you define $f(x)$ on the interval $[1, 3]$ in order to make $f$ continuous on $(-\infty, \infty)$? (A geometric argument will suffice.)

32. Let $f(x) = x + (4/x)$ for $x \leqslant -\frac{1}{2}$ and $x \geqslant 2$. Define $f(x)$ for $x$ in $(-\frac{1}{2}, 2)$ in such a way that the resulting function is continuous on the whole real line.
33. Let $f(x)$ be defined by $f(x) = (x^2 - 1)/(x - 1)$ for $x \neq 1$. How should you define $f(1)$ to make the resulting function continuous? [*Hint:* Plot a graph of $f(x)$ for $x$ near 1 by factoring the numerator.]
34. Let $f(x)$ be defined by $f(x) = 1/x$ for $x \neq 0$. Is there any way to define $f(0)$ so that the resulting function will be continuous?
35. Sketch the graph of a function which is continuous on the whole real line and differentiable everywhere except at $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
36. Is a function which is continuous at $x_0$ necessarily differentiable there? Prove or give a counterexample.
37. The function $f(x) = 1/(x - 1)$ never takes the value zero, yet $f(0) = -1$ is negative and $f(2) = 1$ is positive. Why isn't this a counterexample to the intermediate value theorem?
38. "Prove" that you were once exactly one meter tall.

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In Exercises 39–42, use the method of bisection to find a root of the given function, on the given interval, to the given accuracy.

39. $x^3 - 11$ on $[2, 3]$; accuracy $\frac{1}{100}$.
40. $x^3 + 7$ on $[-3, -1]$; accuracy $\frac{1}{100}$.
41. $x^5 + x^2 + 1$ on $[-2, -1]$; accuracy $\frac{1}{1000}$.
42. $x^4 - 3x - 2$ on $[1, 2]$; accuracy $\frac{1}{500}$.

★43. In the method of bisection, each estimate of the solution of $f(x) = c$ is approximately twice as accurate as the previous one. Examining the list of powers of 2: 2, 4, 8, *16*, 32, 64, *128*, 256, 512, *1024*, 2048, 4096, 8192, *16384*, 32768, 65536, *131072*, . . . —suggests that we get one more decimal place of accuracy for every three or four repetitions of the procedure. Explain.

★44. Using the result of Exercise 43, determine how many times to apply the bisection procedure to guarantee the accuracy $A$ for the interval $[a, b]$ if:
(a) $A = \frac{1}{100}$; $[a, b] = [3, 4]$.
(b) $A = \frac{1}{1000}$; $[a, b] = [-1, 3]$.
(c) $A = \frac{1}{700}$; $[a, b] = [11, 23]$.
(d) $A = \frac{1}{15}$; $[a, b] = [0.1, 0.2]$.

★45. Can you improve the method of bisection by choosing a better point than the one halfway between the two previous points? [*Hint:* If $f(a)$ and $f(b)$ have opposite signs, choose the point where the line through $(a, f(a))$ and $(b, f(b))$ crosses the $x$ axis.] Is there a method of division more appropriate to the decimal system?

Do some experiments to see whether your method gives more accurate answers then the bisection method in the same number of steps. If so, does the extra accuracy justify the extra time involved in carrying out each step? You might wish to compare various methods on a competitive basis, either with friends, with yourself by timing the calculations, or by timing calculations done on a programmable calculator or microcomputer.

★46. Prove that if $f$ is continuous on an interval $I$ (not necessarily closed) and $f(x) \neq 0$ for all $x$ in $I$, then the sign of $f(x)$ is the same for all $x$ in $I$.

★47. Suppose that $f$ is continuous at $x_0$ and that in any open interval $I$ containing $x_0$ there are points $x_1$ and $x_2$ such that $f(x_1) < 0$ and $f(x_2) > 0$. Explain why $f(x_0)$ must equal 0.

★48. (a) Suppose that $f$ and $g$ are continuous on the real line. Show that $f - g$ is continuous. (b) Suppose that $f$ and $g$ are continuous functions on the whole real line. Prove that if $f(x) \neq g(x)$ for all $x$, and $f(0) > g(0)$, then $f(x) > g(x)$ for all $x$. Interpret your answer geometrically.

★49. Prove that any odd-degree polynomial has a root by following these steps.
1. Reduce the problem to showing that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has a root, where $a_i$ are constants and $n$ is odd.
2. Show that if $|x| > 1$ and $|x| > 2 \times \{|a_0| + \cdots + |a_{n-1}|\}$, then $f(x)/x^n$ is positive.
3. Conclude that $f(x) < 0$ if $x$ is large negative and $f(x) > 0$ if $x$ is large positive.
4. Apply the intermediate value theorem.

★50. Show that the polynomial $x^4 + bx + c = 0$ has a (real) root if $256c^3 < 27b^4$. Hint: use the fact that the tangent line is horizontal at the lowest point on the graph.

# 3.2 Increasing and Decreasing Functions

*The sign of the derivative indicates whether a function is increasing or decreasing.*

We begin this section by defining what it means for a function to be increasing or decreasing. Then we show that a function is increasing when its derivative is positive and is decreasing when its derivative is negative. Local maximum and minimum points occur where the derivative changes sign.

We can tell whether a function is increasing or decreasing at $x_0$ by seeing how its graph crosses the horizontal line $y = f(x_0)$ at $x_0$ (see Fig. 3.2.1). This

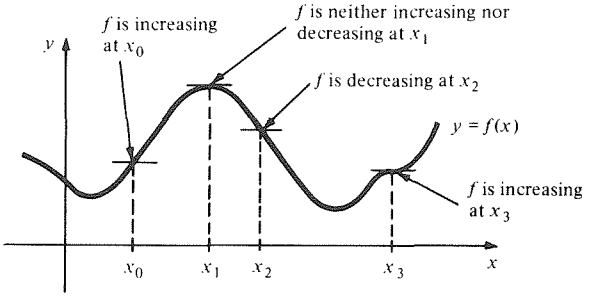


**Figure 3.2.1.** Places where the function $f$ is increasing and decreasing.

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geometric picture is the basis of the precise definition of increasing and decreasing. Note that at the point $x_0$ in Fig. 3.2.1, $f(x)$ is less than $f(x_0)$ for $x$ just to the left of $x_0$, while $f(x)$ is larger than $f(x_0)$ for $x$ just to the right. We cannot take $x$ too far to the right, as the figure shows. The following paragraph gives the technical definition.

We say that a function $f$ is *increasing* at $x_0$ if there is an interval $(a,b)$ containing $x_0$ such that:

1. If $a < x < x_0$, then $f(x) < f(x_0)$.
2. If $x_0 < x < b$, then $f(x) > f(x_0)$.

Similarly, $f$ is *decreasing* at $x_0$ if there is an interval $(a,b)$ containing $x_0$ such that:

1. If $a < x < x_0$, then $f(x) > f(x_0)$.
2. If $x_0 < x < b$, then $f(x) < f(x_0)$.

The purpose of the interval $(a,b)$ is to limit our attention to a small region about $x_0$. Indeed, the notions of increasing and decreasing at $x_0$ are local; they depend only on the behavior of the function near $x_0$. In examples done "by hand," such as Example 1 below, we must actually find the interval $(a,b)$. We will soon see that calculus provides an easier method of determining where a function is increasing and decreasing.

**Example 1** Show that $f(x) = x^2$ is increasing at $x_0 = 2$.

**Solution** Choose $(a,b)$ to be, say, $(1,3)$. If $1 < x < 2$, we have $f(x) = x^2 < 4 = x_0^2$. If $2 < x < 3$, then $f(x) = x^2 > 4 = x_0^2$. We have verified conditions 1 and 2 of the definition, so $f$ is increasing at $x_0 = 2$. ▲

Of special interest is the case $f(x_0) = 0$. If $f$ is increasing at such an $x_0$, we say that $f$ *changes sign from negative to positive at* $x_0$. By definition this occurs when $f(x_0) = 0$ and there is an open interval $(a,b)$ containing $x_0$ such that $f(x) < 0$ when $a < x < x_0$ and $f(x) > 0$ when $x_0 < x < b$. (See Fig. 3.2.2.) Similarly if $f(x_0) = 0$ and $f$ is decreasing at $x_0$, we say that $f$ *changes sign from positive to negative at* $x_0$. (See Fig. 3.2.3.)

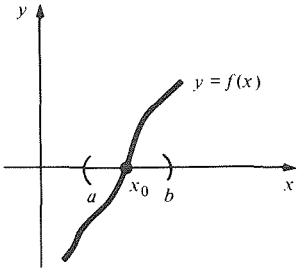


**Figure 3.2.2.** This function changes sign from negative to positive at $x_0$.

Notice that the chosen interval $(a,b)$ may have to be small, since a function which changes sign from negative to positive may later change back from positive to negative (see Fig. 3.2.4).

Changes of sign can be significant in everyday life as, for instance, when the function $b = f(t)$ representing your bank balance changes from positive to negative. Changes of sign will be important for mathematical reasons in the next few sections of this chapter; it will then be useful to be able to determine when a function changes sign by looking at its formula.

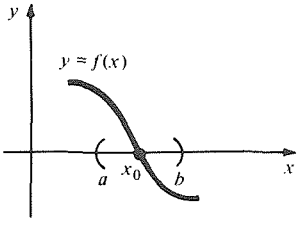


**Figure 3.2.3.** This function changes sign from positive to negative at $x_0$.

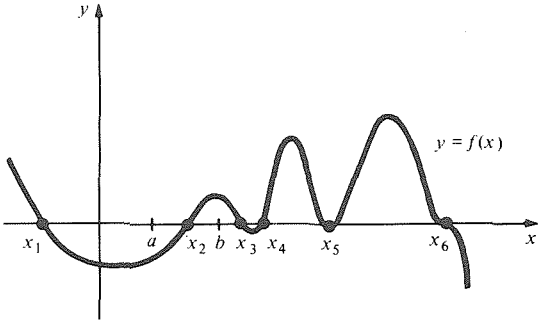


**Figure 3.2.4.** This function changes sign from negative to positive at $x_2$ and $x_4$ and from positive to negative at $x_1$, $x_3$ and $x_6$; it does neither at $x_5$.

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**Example 2** Where does $f(x) = 3x - 5$ change sign?

**Solution** We begin by finding those $x$ for which $f(x)$ is negative and those for which it is positive. First, $f$ is negative when

$$3x - 5 < 0,$$
$$3x < 5,$$
$$x < \tfrac{5}{3}.$$

(If you had difficulty following this argument, you may wish to review the material on inequalities in Section R.1.) Similarly, $f$ is positive when $x > \frac{5}{3}$. So $f$ changes sign from negative to positive at $x = \frac{5}{3}$. (See Fig. 3.2.5.) Here the chosen interval $(a, b)$ can be arbitrarily large. ▲

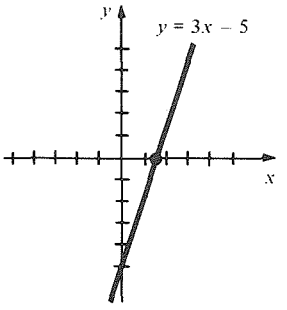


**Figure 3.2.5.** This function changes sign from negative to positive at $x = 5/3$.

If a function is given by a formula which factors, this often helps us to find the changes of sign.

**Example 3** Where does $f(x) = x^2 - 5x + 6$ change sign?

**Solution** We write $f(x) = (x - 3)(x - 2)$. The function $f$ changes sign whenever one of its factors does. This occurs at $x = 2$ and $x = 3$. The factors have opposite signs for $x$ between 2 and 3, and the same sign otherwise, so $f$ changes from positive to negative at $x = 2$ and from negative to positive at $x = 3$. (See Fig. 3.2.6.) ▲

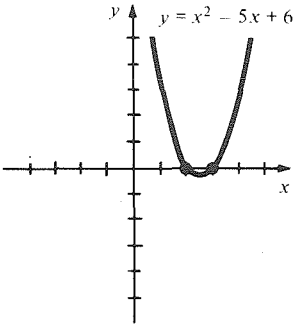


**Figure 3.2.6.** This function changes sign from positive to negative at $x = 2$ and from negative to positive at $x = 3$.

Given a more complicated function, such as $x^5 - x^3 - 2x^2$, it may be difficult to tell directly whether it is increasing or decreasing at a given point. The derivative is a very effective tool for helping us answer such questions.

A linear function $l(x) = mx + b$ is increasing everywhere when the slope $m$ is positive. If the derivative $f'(x)$ of a function $f$ is positive when $x = x_0$, the linear approximation to $f$ at $x_0$ is increasing, so we may expect $f$ to be increasing at $x_0$ as well. (See Fig. 3.2.7.)

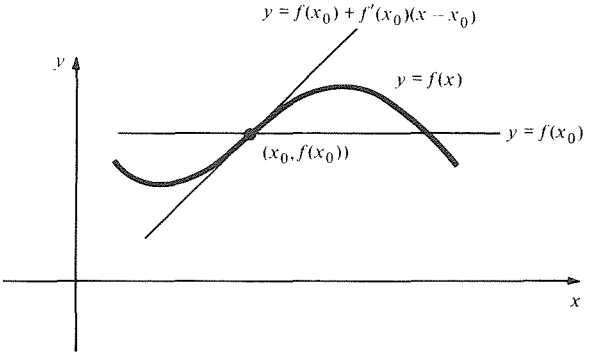


**Figure 3.2.7.** The function $f$ is increasing where its derivative is positive.

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We can verify that a function is increasing where its derivative is positive by using difference quotients. Let $y = f(x)$ and assume that $f'(x_0) > 0$. Then

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} > 0$$

for $\Delta x$ sufficiently small, since $\Delta y/\Delta x$ approaches the positive number $f'(x_0)$ as $\Delta x$ approaches zero. Thus, there is an interval $(a,b)$ about $x_0$ such that $\Delta x = x - x_0$ and $\Delta y = f(x) - f(x_0)$ have the same sign when $x$ is in the interval. Thus, when $a < x < x_0$, $\Delta x$ is negative and so is $\Delta y$; hence $f(x) < f(x_0)$. Similarly, $f(x) > f(x_0)$ when $x_0 < x < b$. Thus $f$ is increasing at $x_0$. If $f'(x_0) < 0$, a similar argument shows that $f$ is decreasing at $x_0$.

The functions $y = x^3$, $y = -x^3$, and $y = x^2$ show that many kinds of behavior are possible when $f'(x_0) = 0$ (see Fig. 3.2.8).

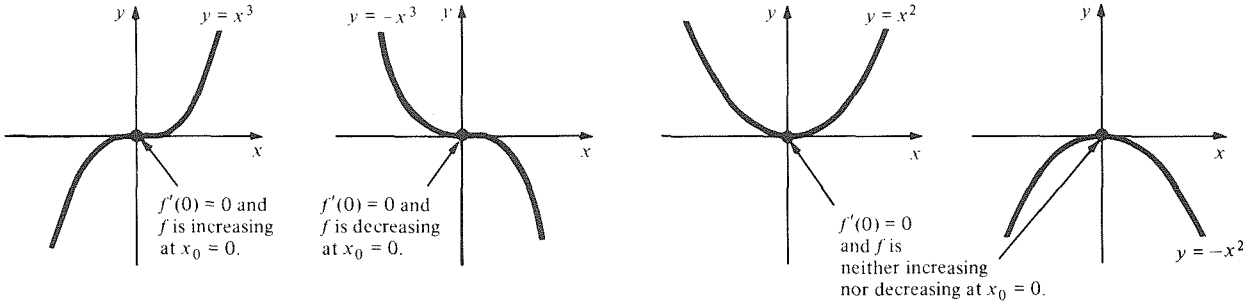
y
y = x³
x
f'(0) = 0 and f is increasing at x₀ = 0.
y = −x³
f'(0) = 0 and f is decreasing at x₀ = 0.
y = x²
f'(0) = 0 and f is neither increasing nor decreasing at x₀ = 0.
y = −x²

**Figure 3.2.8.** If $f'(x_0) = 0$, you cannot tell if $f$ is increasing or decreasing without further information.

Thus we arrive at the following test (see Fig. 3.2.9).

**Increasing–Decreasing Test**

1. If $f'(x_0) > 0$, $f$ is increasing at $x_0$.
2. If $f'(x_0) < 0$, $f$ is decreasing at $x_0$.
3. If $f'(x_0) = 0$, the test is inconclusive.

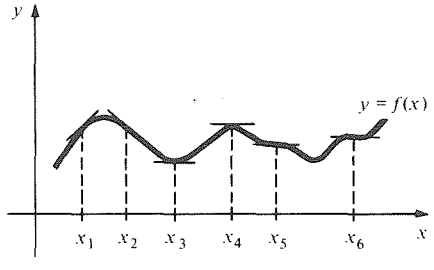


**Figure 3.2.9.** $f'(x_1) > 0$; $f$ is increasing at $x_1$. $f'(x_2) < 0$; $f$ is decreasing at $x_2$. $f'(x_3) = f'(x_4) = 0$; $f$ is neither increasing nor decreasing at $x_3$ and $x_4$. $f'(x_5) = 0$; $f$ is decreasing at $x_5$. $f'(x_6) = 0$; $f$ is increasing at $x_6$.

**Example 4** (a) Is $x^5 - x^3 - 2x^2$ increasing or decreasing at $-2$? (b) Is $g(s) = \sqrt{s^2 - s}$ increasing or decreasing at $s = 2$?

**Solution** (a) Letting $f(x) = x^5 - x^3 - 2x^2$, we have $f'(x) = 5x^4 - 3x^2 - 4x$, and $f'(-2) = 5(-2)^4 - 3(-2)^2 - 4(-2) = 80 - 12 + 8 = 76$, which is positive. Thus $x^5 - x^3 - 2x^2$ is increasing at $-2$.

(b) By the chain rule, $g'(s) = (2s - 1)/2\sqrt{s^2 - s}$, so $g'(2) = 3/2\sqrt{2} > 0$. Thus $g$ is increasing at $s = 2$. ▲

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**Example 5** Let $f(x) = x^3 - x^2 + x + 3$. How does $f$ change sign at $x = -1$?

**Solution** Notice that $f(-1) = 0$. Also, $f'(-1) = 3(-1)^2 - 2(-1) + 1 = 6 > 0$, so $f$ is increasing at $x = -1$. Thus $f$ changes sign from negative to positive. ▲

We can also interpret the increasing–decreasing test in terms of velocities and other rates of change. For instance, if $f(t)$ is the position of a particle on the real-number line at time $t$, and $f'(t_0) > 0$, then the particle is moving to the right at time $t_0$; if $f'(t_0) < 0$, the particle is moving to the left (see Fig. 3.2.10).

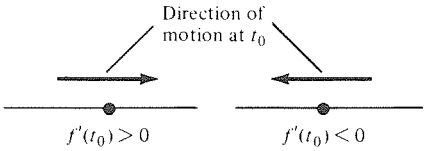


**Figure 3.2.10.** Positive velocity means that the motion is to the right, and negative velocity means that the motion is to the left.

**Example 6** The temperature at time $t$ is given by $f(t) = (t + 1)/(t - 1)$ for $t < 1$. Is it getting warmer or colder at $t = 0$?

**Solution** We calculate $f'(t)$ by the quotient rule:

$$f'(t) = \frac{(t-1) - (t+1)}{(t-1)^2} = -\frac{2}{(t-1)^2}.$$

Since $f'(0) = -2$ is negative, $f$ is decreasing, so it is getting colder. ▲

Instead of focusing our attention on the small intervals used so far, one can also consider the idea of increasing and decreasing functions on general intervals, which could be large.

Let $f$ be a function defined on an interval $I$. If $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in $I$, we say that $f$ is *increasing on* $I$. If $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in $I$, we say that $f$ is *decreasing on* $I$.

It is plausible that *if $f$ is increasing at each point of an interval, then $f$ is increasing on the whole interval* in this new sense. We shall use this important fact now, deferring the formal proof until Section 3.6.

**Example 7** On what intervals is $f(x) = x^3 - 2x + 6$ increasing or decreasing?

**Solution** We consider the derivative $f'(x) = 3x^2 - 2$. This is positive when $3x^2 - 2 > 0$, i.e., when $x^2 > 2/3$, i.e., either $x > \sqrt{2/3}$ or $x < -\sqrt{2/3}$. Similarly, $f'(x) < 0$ when $x^2 < 2/3$, i.e., $-\sqrt{2/3} < x < \sqrt{2/3}$. Thus, $f$ is increasing on the intervals $(-\infty, -\sqrt{2/3})$ and $(\sqrt{2/3}, \infty)$, and $f$ is decreasing on $(-\sqrt{2/3}, \sqrt{2/3})$. ▲

The result of Example 7 enables us to make a good guess at what the graph $y = f(x)$ looks like. We first plot the points where $x = \pm\sqrt{2/3}$ as in Fig. 3.2.11(a). When $x = \pm\sqrt{2/3}$, we get $y = 6 \mp 4\sqrt{6}/9$. We also plot the point $x = 0$, $y = 6$. Since $f$ is increasing on $(-\infty, -\sqrt{2/3})$ and on $(\sqrt{2/3}, \infty)$, and decreasing elsewhere, the graph must look something like that in Fig. 3.2.11(b). Later in this chapter, we will use techniques like this to study graphing more systematically.

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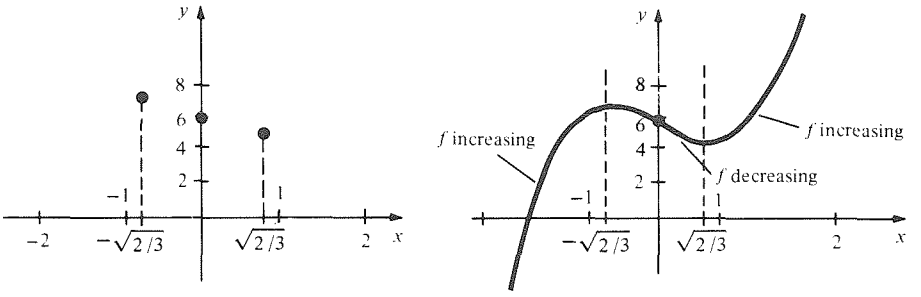


(a) These points lie on the graph $y = x^3 - 2x + 6$.

(b) We draw $f$ increasing on $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$, on $\left(\sqrt{\frac{2}{3}}, \infty\right)$ and decreasing on $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

**Figure 3.2.11.** First steps in sketching a graph.

**Example 8** Match each of the functions in the left-hand column of Fig. 3.2.12 with its derivative in the right-hand column.

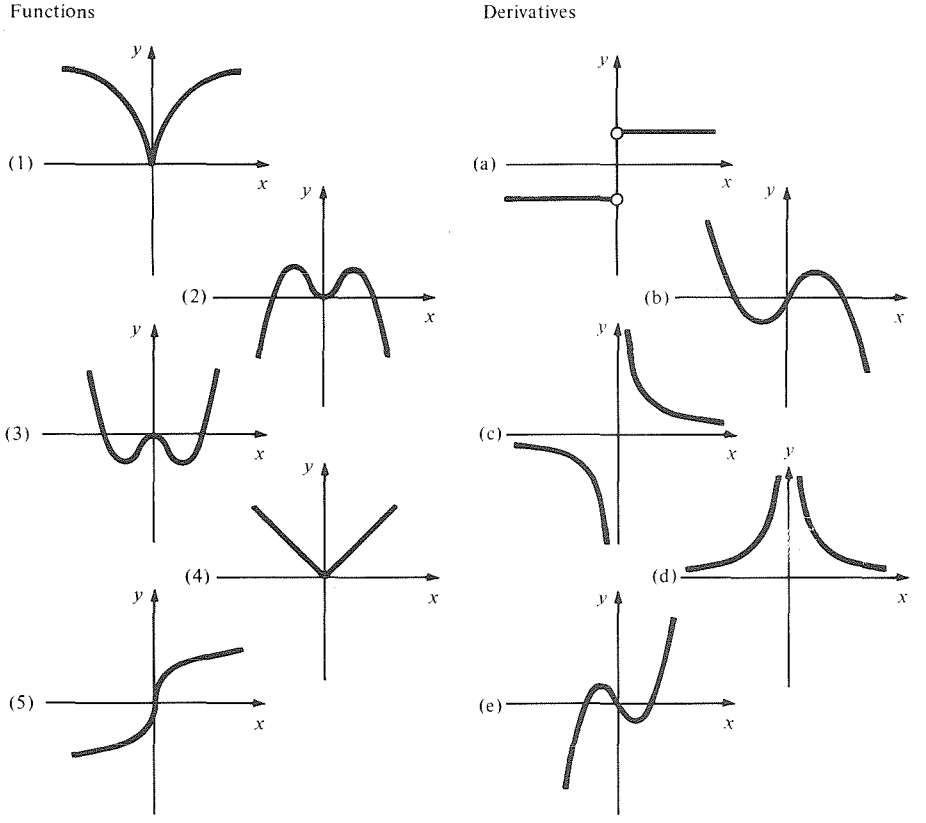


**Figure 3.2.12.** Matching functions and their derivatives.

**Solution** Function (1) is decreasing for $x < 0$ and increasing for $x > 0$. The only functions in the right-hand column which are negative for $x < 0$ and positive for $x > 0$ are (a) and (c). We notice, further, that the derivative of function 1 is not constant for $x < 0$ (the slope of the tangent is constantly changing), which eliminates (a). Similar reasoning leads to the rest of the answers, which are: (1)-(c), (2)-(b), (3)-(e), (4)-(a), (5)-(d). ▲

We now turn our attention to the points which separate the intervals on which a function is increasing or decreasing, such as the points $x = \pm\sqrt{2/3}$ in Fig.

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3.2.11. In this figure, we see that these points are places where $f$ has a maximum and a minimum. Here are the formal definitions: A point $x_0$ is called a *local minimum point* for $f$ if there is an open interval $(a, b)$ around $x_0$ such that $f(x) \geq f(x_0)$ for all $x$ in $(a, b)$. Similarly, a point $x_0$ is called a *local maximum point* for $f$ if there is an open interval $(a, b)$ around $x_0$ such that $f(x) \leq f(x_0)$ for all $x$ in $(a, b)$.[2] (See Fig. 3.2.13.)

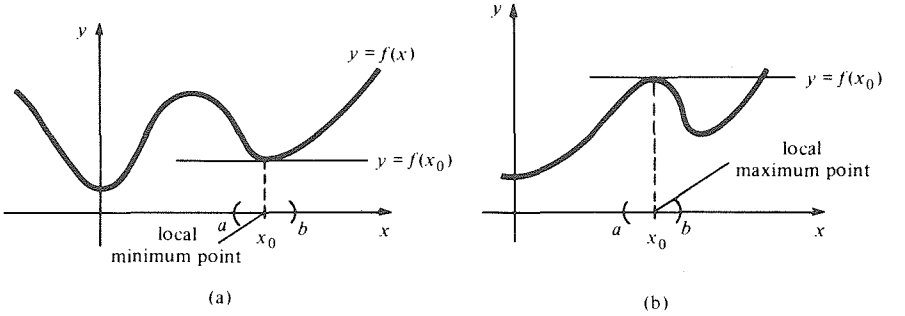


**Figure 3.2.13.** The graph of $f$ lies above the line $y = f(x_0)$ near a local minimum point (a) and below that line near a local maximum point (b).

**Example 9** For each of the functions in Fig. 3.2.14, tell whether $x_0$ is a local minimum point, a local maximum point, or neither.

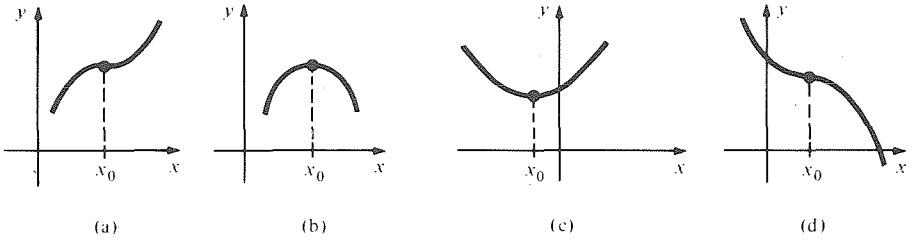


**Figure 3.2.14.** Is $x_0$ a local maximum? Local minimum? Neither?

**Solution** Comparing each graph in Fig. 3.2.14 with the horizontal line through the heavy dot, we find that $x_0$ is a local maximum point in (b), a local minimum point in (c), and neither in (a) and (d). ▲

At a local maximum or minimum point $x_0$, a function $f$ can be neither increasing nor decreasing, as we see from a comparison of the definitions or graphic interpretations of these concepts. It follows that the derivative $f'(x_0)$ (if it exists) can be neither positive nor negative at such a point; hence, it must be zero. Points $x_0$ where $f'(x_0) = 0$ are called *critical points* of $f$.

The critical point test described in the following display is very important. (Some people remember nothing else after a year of calculus.) A good portion of this chapter explores the applications of the test and its limitations.

**Critical Point Test**

If $x_0$ is a local maximum or minimum point of a (differentiable) function $f$, then $x_0$ is a critical point, i.e., $f'(x_0) = 0$.

In Figs. 3.2.8 and 3.2.9, you will observe that not every critical point is a local maximum or minimum. In the remainder of this section we shall develop a test for critical points to be maxima or minima. This will be useful for both

[2] Sometimes the phrase "strict local minimum point" is used when $f(x) > f(x_0)$ for all $x$ in $(a, b)$ other than $x_0$. Likewise for strict local maximum points. Here, and elsewhere, we use the term "point" to refer to a number in the domain of $f$ rather than to a point in the plane.

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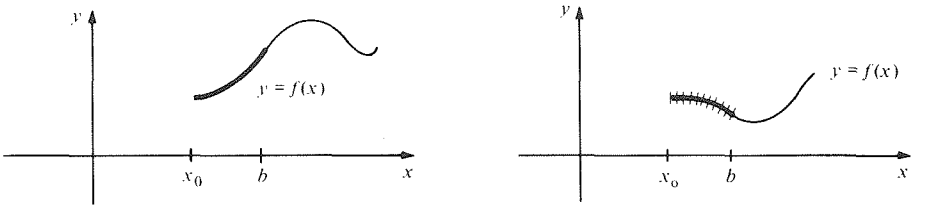
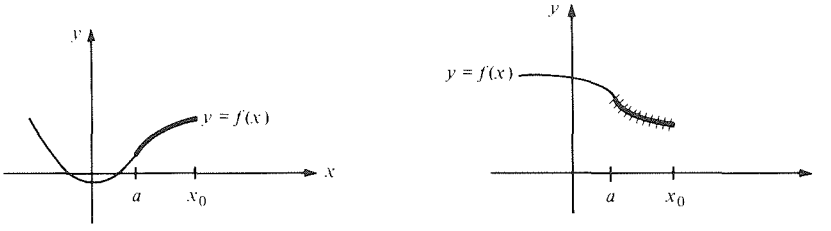
graphing and problem solving. To lead to the test, we ask this question: How does a function $f$ behave just to the right of a critical point $x_0$? The two simplest possibilities are that $f'(x) > 0$ for all $x$ in some interval $(x_0, b)$ or that $f'(x) < 0$ in such an interval. In the first case, $f$ is increasing on $(x_0, b)$, and the second case is decreasing there. These possibilities are illustrated in Fig. 3.2.15.




$f'(x) > 0$ on $(x_0, b)$ $\qquad$ $f'(x) < 0$ on $(x_0, b)$

**Figure 3.2.15.** Behavior to the right of a critical point.

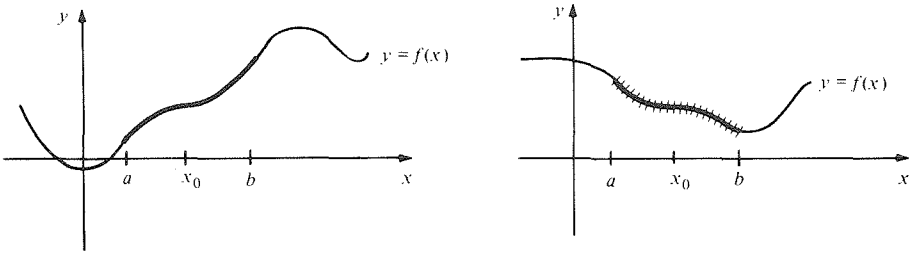
Likewise, the behavior of $f$ just to the left of $x_0$ can be determined if we know the sign of $f'(x)$ on an interval $(a, x_0)$, as shown in Fig. 3.2.16.




$f'(x) > 0$ on $(a, x_0)$ $\qquad$ $f'(x) < 0$ on $(a, x_0)$

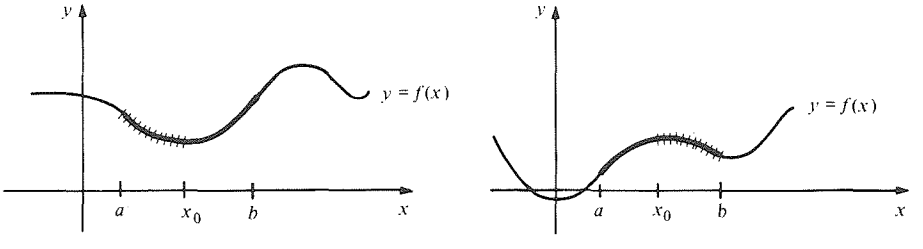
**Figure 3.2.16.** Behavior to the left of a critical point.

The two possibilities in Fig. 3.2.15 can be put together with the two possibilities in Fig. 3.2.16 to give four different ways in which a function may behave near a critical point, as shown in Fig. 3.2.17.




$f'(x) > 0$ on $(a, x_0)$
$f'(x) > 0$ on $(x_0, b)$
$f$ is increasing at $x_0$

$f'(x) < 0$ on $(a, x_0)$
$f'(x) < 0$ on $(x_0, b)$
$f$ *is* decreasing at $x_0$




$f'(x) < 0$ on $(a, x_0)$
$f'(x) > 0$ on $(x_0, b)$
$x_0$ is a local minimum point

$f'(x) > 0$ on $(a, x_0)$
$f'(x) < 0$ on $(x_0, b)$
$x_0$ is a local maximum point

**Figure 3.2.17.** Four ways in which a function can behave near a critical point.

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The situations shown in Fig. 3.2.17 are not the only possibilities, but they are the most common. For example, $f'(x)$ might be zero on one side of $x_0$ or the other, or it might oscillate wildly. For most of the functions we encounter in this course, though, the classification of critical points in Fig. 3.2.17 will suffice. As we have already seen in Fig. 3.2.8, the functions $x^3$, $-x^3$, $x^2$, and $-x^2$ provide examples of all four possibilities. Notice that in the case where $x_0$ is a local maximum or minimum point, $f'$ changes sign at $x_0$, while $f'$ has the same sign on both sides of a critical point where $f$ is increasing or decreasing.

We summarize our analysis of critical points in the form of a test.

## First Derivative Test

Suppose that $x_0$ is a critical point for $f(x)$, i.e., $f'(x_0) = 0$.

1. If $f'(x)$ changes sign from negative to positive at $x_0$, then $x_0$ is a (strict) local minimum point for $f$.
2. If $f'(x)$ changes sign from positive to negative at $x_0$, then $x_0$ is a (strict) local maximum point for $f$.
3. If $f'(x)$ is negative for all $x \neq x_0$ near $x_0$, then $f$ is decreasing at $x_0$.
4. If $f'(x)$ is positive for all $x \neq x_0$ near $x_0$, then $f$ is increasing at $x_0$.

| Sign of $f'(x)$ near $x_0$ to the left | Sign of $f'(x)$ near $x_0$ to the right | Behavior of $f$ at $x_0$ |
|---|---|---|
| $-$ | $+$ | local minimum |
| $+$ | $-$ | local maximum |
| $-$ | $-$ | decreasing |
| $+$ | $+$ | increasing |

This test should *not* be literally memorized. If you understand Fig. 3.2.17, you can reproduce the test accurately.

**Example 10** Find the critical points of the function $f(x) = 3x^4 - 8x^3 + 6x^2 - 1$. Are they local maximum or minimum points?

**Solution** We begin by finding the critical points: $f'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1) = 12x(x - 1)^2$; the critical points are thus 0 and 1. Since $(x - 1)^2$ is always nonnegative, the only sign change is from negative to positive at 0. Thus 0 is a local minimum point, and $f$ is increasing at 1. ▲

**Example 11** Find and classify the critical points of the function

$$f(x) = x^3 + 3x^2 - 6x.$$

**Solution** The derivative $f'(x) = 3x^2 + 6x - 6$ has roots at $-1 \pm \sqrt{3}$; it is positive on $(-\infty, -1 - \sqrt{3})$ and $(-1 + \sqrt{3}, \infty)$ and is negative on $(-1 - \sqrt{3}, -1 + \sqrt{3})$. Changes of sign occur at $-1 - \sqrt{3}$ (positive to negative) and $-1 + \sqrt{3}$ (negative to positive), so $-1 - \sqrt{3}$ is a local maximum point and $-1 + \sqrt{3}$ is a local minimum point. ▲

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**Example 12** Discuss the critical points of $y = x^4$ and $y = -x^4$.

**Solution** If $f(x) = x^4$, $f'(x) = 4x^3$, and the only critical point is at $x_0 = 0$. We know that $x^3$ changes sign from negative to positive at 0, so the same is true for $4x^3$, and hence $x^4$ has a local minimum at 0. Similarly, the only critical point of $-x^4$ is 0, which is a local maximum point. (See Fig. 3.2.18.) ▲

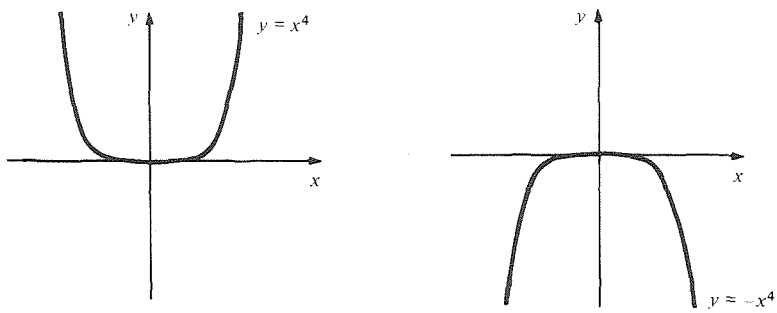


**Figure 3.2.18.** Critical behavior of $y = x^4$ and $y = -x^4$.

## Exercises for Section 3.2

1. Using algebra alone, show that $f(x) = x^2$ is increasing at $x_0 = 3$.
2. Using algebra alone, show that $f(x) = x^2$ is decreasing at $x_0 = -1$.
3. Show by algebra that $f(x) = mx + b$ is increasing for all $x_0$ if $m > 0$.
4. Show by algebra that $f(x) = mx + b$ is decreasing for all $x_0$ if $m < 0$.

Using only algebra, find the sign changes of the functions at the indicated points in Exercises 5–8.

5. $f(x) = 2x - 1$; $x_0 = \frac{1}{2}$.
6. $f(x) = x^2 - 1$; $x_0 = -1$.
7. $f(x) = x^5$; $x_0 = 0$.
8. $h(z) = z(z - 2)$; $z_0 = 2$.

In Exercises 9–12, determine whether the functions are increasing, decreasing, or neither at the indicated points.

9. $x^3 + x + 1$; $x_0 = 0$.
10. $x^4 + x + 5$; $x_0 = 0$.
11. $\dfrac{x^2 - 1}{\sqrt{x^2 + 1}}$; $x_0 = 0$.
12. $\dfrac{1}{(x^4 - x^3 + 1)^{3/2}}$; $x_0 = 1$.
13. If $f(t) = t^5 - t^4 + 2t^2$ is the position of a particle on the real-number line at time $t$, is it moving to the left or right at $t = 1$?
14. A ball is thrown upward with an initial velocity of 30 meters per second. The ball's height above the ground at time $t$ is $h(t) = 30t - 4.9t^2$. When is the ball rising? When is it falling?
15. The annual inflation rate in Uland during 1968 was approximately $r(t) = 20[1 + (t^2 - 6t)/500]$ percent *per year*, where $t$ is the time *in months* from the beginning of the year. During what months was the inflation rate decreasing? What are the max–min points of $r(t)$? Explain their (political) significance.

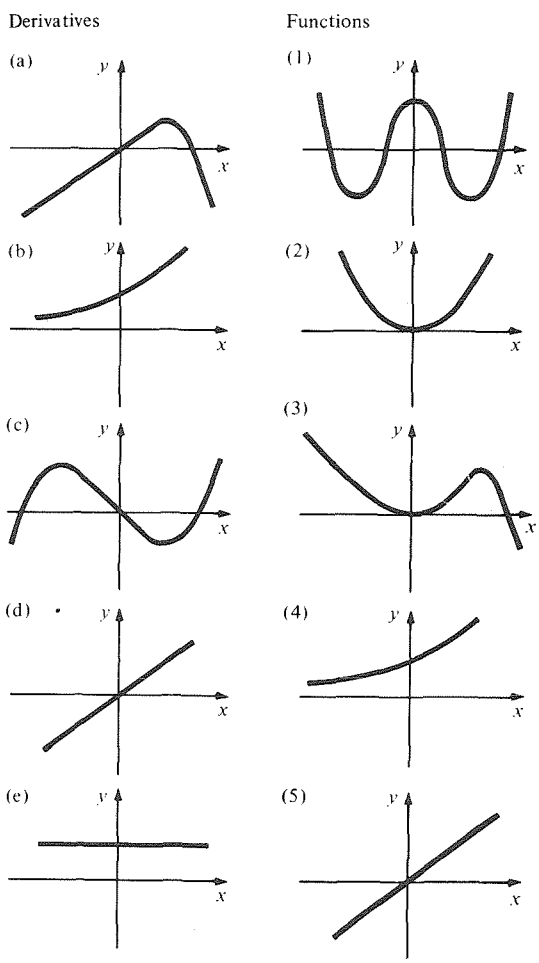


**Figure 3.2.19.** Matching derivatives and functions (Exercise 21).

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16. The rate of a chemical reaction between $t = 0$ and $t = 10$ is given by $r(t) = 2t^3 - 3t^2 + 1$. When is the reaction slowing down? When is it speeding up?
17. Find the intervals on which $f(x) = x^2 - 1$ is increasing or decreasing.
18. Find the intervals on which $x^3 - 3x^2 + 2x$ is increasing or decreasing.
19. Find the points at which $f(x) = 2x^3 - 9x^2 + 12x + 5$ is increasing or decreasing.
20. Find all points in which $f(x) = x^2 - 3x + 2$ is increasing, and at which it changes sign.
21. Match each derivative on the left in Fig. 3.2.19 with the function on the right.
22. Sketch functions whose derivatives are shown in Fig. 3.2.20.

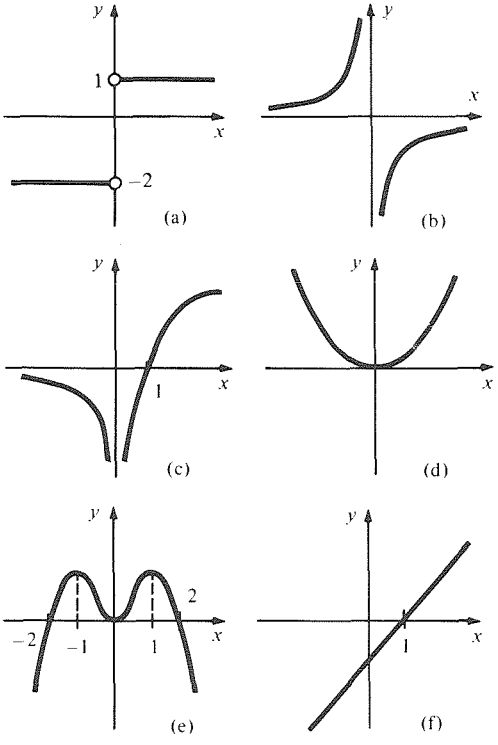


**Figure 3.2.20.** Sketch functions that have these derivatives.

23. For each of the functions shown in Fig. 3.2.20 state: (1) where it is increasing: (2) where it is decreasing; (3) its local maximum and minimum points; (4) where it changes sign.
24. For each of the functions in Fig. 3.2.21 tell whether the function is increasing or decreasing and whether the derivative of the function is increasing or decreasing.

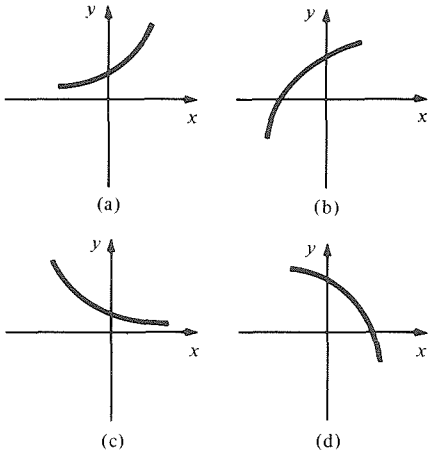


**Figure 3.2.21.** Are $f$ and $f'$ increasing or decreasing?

25. In Fig. 3.2.22, which points are local maxima and which are local minima?

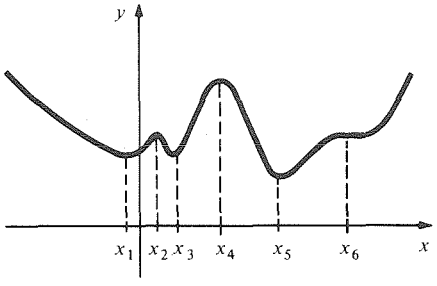


**Figure 3.2.22.** Find the local maxima and minima.

26. Tell where the function in Fig. 3.2.22 is increasing and where it is decreasing.

Find the critical points of the functions in Exercises 27–34 and decide whether they are local maxima, local minima, or neither.

27. $f(x) = x^2 - 2$
28. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$
29. $f(x) = x^3 + x^2 - 2$
30. $f(x) = 3x^4$
31. $g(y) = \dfrac{y}{1 + y^2}$
32. $h(z) = \dfrac{z^2}{1 + z^2}$
33. $l(r) = (r^4 - r^2)^2$
34. $m(q) = (q^3 - q)^{347}$
35. Is $f(x) = 1/(x^2 + 1)$ increasing or decreasing at $x = 1, -3, \frac{3}{4}, 25, -36$?
36. Let $f(x) = 4x^2 + (1/x)$. Determine whether $f$ is increasing or decreasing at each of the following points: (a) 1; (b) $-\frac{1}{2}$; (c) $-5$; (d) $2\frac{1}{3}$.
37. Describe the change of sign at $x = 0$ of the function $f(x) = mx$ for $m = -2, 0, 2$.
38. Describe the change of sign at $x = 0$ of the function $f(x) = mx - x^2$ for $m = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$.

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Find the sign changes (if any) for each of the functions in Exercises 39–42.

39. $x^2 + 3x + 2$
40. $x^2 + x - 1$
41. $x^2 - 4x + 4$
42. $\dfrac{x^3 - 1}{x^2 + 1}$
43. Using only algebra, determine the sign change of $f(x) = (x - r_1)(x - r_2)$ at $x = r_1$, where $r_1 < r_2$.
44. For an observer standing on the Earth, let $f(t)$ denote the angle from the horizon line to the sun at time $t$. When does $f(t)$ change sign?

Find the intervals on which each of the functions in Exercises 45–48 is increasing or decreasing:

45. $2x^3 - 5x + 7$
46. $x^5 - x^3$
47. $\dfrac{x^2 + 1}{x - 3}$
48. $x^4 - 2x^2 + 1$
49. Find a quadratic polynomial which is zero at $x = 1$, is decreasing if $x < 2$, and is increasing if $x > 2$.
50. Herring production $T$ (in grams) is related to the number $N$ of fish stocked in a storage tank by the equation $T = 500N - 50N^2$.
    (a) Find $dT/dN$.
    (b) Unless too many fish are stocked, an increase in the number of fish stocked will cause an increase in production at the expense of a reduction in the growth of each fish. (The weight for each fish is $T/N$.) Explain this statement mathematically in terms of derivatives and the level $N^*$ of stocking which corresponds to maximum production.

★51. Prove the following assertions concerning the function $f(x) = (x^3 - 1)/(x^2 - 1)$:
    (a) $f$ can be defined at $x = 1$ so that $f$ becomes continuous and differentiable there, but cannot be so defined at $x = -1$.
    (b) $f$ is increasing on $(-\infty, -2]$ and decreasing on $[-2, -1)$.
    (c) If $a < -1$, then $(a^3 - 1)/(a^2 - 1) \leqslant -3$.
    (d) $f$ is increasing on $[0, \infty)$ and decreasing on $(-1, 0]$. Make up an equality based on this fact.

★52. Prove that $f$ is increasing at $x_0$ if and only if the function $f(x) - f(x_0)$ changes sign from negative to positive at $x_0$.

★53. Using the definition of an increasing function, prove that if $f$ and $g$ are increasing at $x_0$, then so is $f + g$.

★54. Prove that if $f$ and $g$ are increasing and positive on an interval $I$, then $fg$ is increasing on $I$.

★55. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Under what conditions on the $a_i$'s is $f$ increasing at $x_0 = 0$?

★56. Under what conditions on $a$, $b$, $c$, and $d$ is the cubic polynomial $ax^3 + bx^2 + cx + d$ strictly increasing or strictly decreasing on $(-\infty, \infty)$? (Assume $a \neq 0$.)

★57. If $g$ and $h$ are positive functions, find criteria involving $g'(x)/g(x)$ and $h'(x)/h(x)$ to tell when (a) the product $g(x)h(x)$ and (b) the quotient $g(x)/h(x)$ are increasing or decreasing.

★58. Let $f$ be a function, and $a > 0$ a positive real number. Discuss the relation between the critical points of $f(x)$, $af(x)$, $a + f(x)$, $f(ax)$, and $f(a + x)$.

★59. Find a cubic polynomial with a graph like the one shown in Fig. 3.2.23.

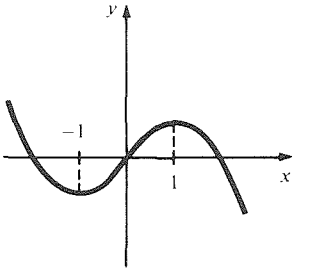


**Figure 3.2.23.** This is the graph of what cubic polynomial?

★60. (a) Show that there is no quartic polynomial whose graph is consistent with the information shown in Fig. 3.2.20(e).
    (b) Show that if $\pm 2$ is replaced by $\pm\sqrt{2}$, then there is a quartic polynomial consistent with the information in Fig. 3.2.20(e).

★61. Find a relationship between the (positive) values of $a$ and $b$ which insures that there is a quartic polynomial with a graph consistent with the information in Fig. 3.2.24.

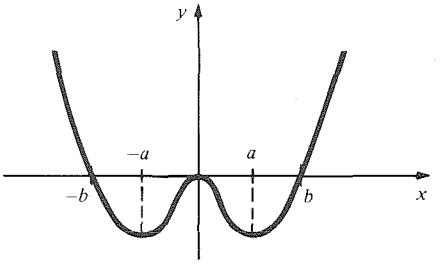


**Figure 3.2.24.** When is this the graph of a quartic?

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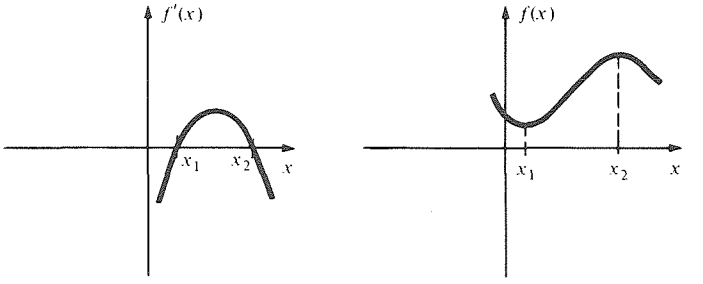
# 3.3 The Second Derivative and Concavity

*The sign of the second derivative indicates which way the graph of a function is bending.*

In the last section we saw that the classification of critical points of a function $f(x)$ depends on the sign changes of the derivative $f'(x)$. On the other hand, the sign changes of $f'(x)$ at a critical point of $f$ are determined by the sign of the derivative of $f'(x)$, i.e., by the sign of the *second* derivative $f''(x)$. (See Fig. 3.3.1.) After exploring the consequences of this idea, we shall see that the sign of $f''(x)$ is important even when $x$ is not a critical point of $f$.




(a) Graph of $f'$ (b) Graph of $f$

**Figure 3.3.1.** $f''(x_1) > 0$ so $f'$ is increasing at $x_1$ and changes sign from negative to positive; thus $f$ has a local minimum at $x_1$. Likewise $f''(x_2) < 0$ and $x_2$ is a local maximum.

Recall that if $g(x_0) = 0$ and $g'(x_0) > 0$, then $g(x)$ changes sign from negative to positive. Applying this to the case where $g$ is the derivative $f'$ of a function $f$, we find that for a critical point $x_0$ of $f(x)$, $f'(x)$ changes sign from negative to positive if $f''(x_0) > 0$. Thus, by the first derivative test, $x_0$ is a local minimum point. Similar reasoning when $f''(x_0) < 0$ leads to the following test.

## Second Derivative Test

Suppose that $f'(x_0) = 0$.

1. If $f''(x_0) > 0$, then $x_0$ is a (strict) local minimum point.
2. If $f''(x_0) < 0$, then $x_0$ is a (strict) local maximum point.
3. If $f''(x_0) = 0$, the test is inconclusive.

We will discuss the case $f''(x_0) = 0$ shortly. For now, notice that the functions $y = x^3$, $y = -x^3$, $y = x^4$, $y = -x^4$ (Figs. 3.2.8 and 3.2.18) show that various things can happen in this case.

**Example 1** Use the second derivative test to analyze the critical points of the function $f(x) = x^3 - 6x^2 + 10$.

**Solution** Since $f'(x) = 3x^2 - 12x = 3x(x - 4)$, the critical points are at 0 and 4. Since $f''(x) = 6x - 12$, we find that $f''(0) = -12 < 0$ and $f''(4) = 12 > 0$. By the second derivative test, 0 is a local maximum point and 4 is a local minimum point. ▲

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**Example 2** Analyze the critical points of $f(x) = x^3 - x$.

**Solution** $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$. The critical points are zeros of $f'(x)$; that is, $x = \pm(1/\sqrt{3})$; $f''(-1/\sqrt{3}) = -(6/\sqrt{3}) < 0$ and $f''(1/\sqrt{3}) = 6/\sqrt{3} > 0$. By the second derivative test, $-(1/\sqrt{3})$ is a local maximum point and $1/\sqrt{3}$ is a local minimum point. ▲

When $f''(x_0) = 0$, the second derivative test is inconclusive. We may sometimes use the first derivative test to analyze the critical points, however.

**Example 3** Analyze the critical point $x_0 = -1$ of $f(x) = 2x^4 + 8x^3 + 12x^2 + 8x + 7$.

**Solution** The derivative is $f'(x) = 8x^3 + 24x^2 + 24x + 8$, and $f'(-1) = -8 + 24 - 24 + 8 = 0$, so $-1$ is a critical point. Now $f''(x) = 24x^2 + 48x + 24$, so $f''(-1) = 24 - 48 + 24 = 0$, and the second derivative test is inconclusive. If we factor $f'$, we find $f'(x) = 8(x^3 + 3x^2 + 3x + 1) = 8(x + 1)^3$. Thus $-1$ is the only root of $f'$, $f'(-2) = -8$, and $f'(0) = 8$, so $f'$ changes sign from negative to positive at $-1$; hence, $-1$ is a local minimum point for $f$. ▲

Whether or not $f'(x_0)$ is zero, the sign of $f''(x_0)$ has an important geometric interpretation: it tells us which way the tangent line to the graph of $f$ turns as the point of tangency moves to the right along the graph (see Fig. 3.3.2). The two graphs in Fig. 3.3.2 are bent in opposite directions. The graph in part (a) is said to be *concave upward*; the graph in part (b) is said to be *concave downward*.

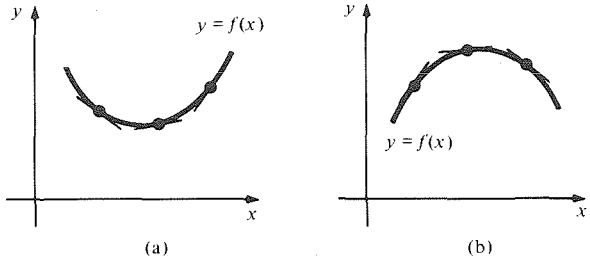


**Figure 3.3.2.** (a) The slope of the tangent line is increasing; $f''(x) > 0$. (b) The slope of the tangent line is decreasing; $f''(x) < 0$.

We can give precise definitions of upward and downward concavity by considering how the graph of $f$ lies in relation to one of its tangent lines. To accomplish this, we compare $f(x)$ with its linear approximation at $x_0$: $l(x) = f(x_0) + f'(x_0)(x - x_0)$. If there is an open interval $(a, b)$ about $x_0$ such that $f(x) > l(x)$ for all $x$ in $(a, b)$ other than $x_0$, then $f$ is called *concave upward at* $x_0$. If, on the other hand, there is an open interval $(a, b)$ about $x_0$ such that $f(x) < l(x)$ for all $x$ in $(a, b)$ other than $x_0$, then $f$ is called *concave downward at* $x_0$.

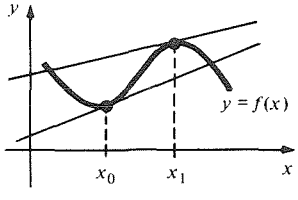


**Figure 3.3.3.** The function $f$ is concave upward at $x_0$ and concave downward at $x_1$.

Geometrically, $f$ is concave upward (downward) at $x_0$ if the graph of $f$ lies locally above (below) its tangent line at $x_0$, as in Fig. 3.3.3.

Notice that the difference $h(x) = f(x) - l(x)$ is positive or negative according to whether $f(x) > l(x)$ or $f(x) < l(x)$. Since $h(x_0) = f(x_0) - l(x_0) = 0$, we see that $f$ is concave upward or downward at $x_0$ according to whether $h(x)$ has a local minimum or maximum at $x_0$. (See Fig. 3.3.4.)

If we differentiate $h(x) = f(x) - [f(x_0) + f'(x_0)(x - x_0)]$ twice, we obtain $h'(x) = f'(x) - f'(x_0)$ and $h''(x) = f''(x)$ ($x_0$ is treated as a constant). Notice that $h'(x_0) = 0$, so $x_0$ is a critical point for $h$. Next, observe that $h''(x_0) = f''(x_0)$, so we may conclude from the second derivative test for local maxima and minima that $x_0$ is a local minimum for $h$ if $f''(x_0) > 0$ and a local maximum if $f''(x_0) < 0$. Thus we have the test in the next box. (Once again, the functions $x^3, -x^3, x^4, -x^4$ at $x_0 = 0$ illustrate the possibilties in case 3.)

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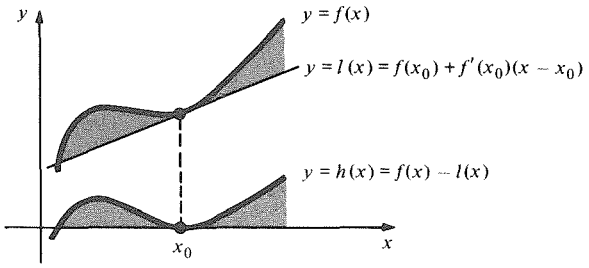


**Figure 3.3.4.** The function $f$ is concave upward at $x_0$ when the difference $h(x)$ between $f(x)$ and its linear approximation at $x_0$ has a local minimum at $x_0$.

## The Second Derivative Test for Concavity

1. If $f''(x_0) > 0$, then $f$ is concave upward at $x_0$.
2. If $f''(x_0) < 0$, then $f$ is concave downward at $x_0$.
3. If $f''(x_0) = 0$, then $f$ may be either concave upward at $x_0$, concave downward at $x_0$, or neither.

**Example 4** Discuss the concavity of $f(x) = 4x^3$ at the points $x = -1$, $x = 0$, and $x = 1$.

**Solution** We have $f'(x) = 12x^2$ and $f''(x) = 24x$, so $f''(-1) = -24$, $f''(0) = 0$, and $f''(1) = 24$. Therefore, $f$ is concave downward at $-1$ and concave upward at 1. At zero the test is inconclusive; we can see, however, that $f$ is neither concave upward nor downward by noticing that $f$ is increasing at zero, so that it crosses its tangent line at zero (the $x$ axis). That is, $f$ is neither above nor below its tangent line near zero, so $f$ is neither concave up nor concave down at zero. ▲

**Example 5** Find the intervals on which $f(x) = 3x^3 - 8x + 12$ is concave upward and on which it is concave downward. Make a rough sketch of the graph.

**Solution** Differentiating $f$, we get $f'(x) = 9x^2 - 8$, $f''(x) = 18x$. Thus $f$ is concave upward when $18x > 0$ (that is, when $x > 0$) and concave downward when $x < 0$. The critical points occur when $f'(x) = 0$, i.e., at $x = \pm\sqrt{8/9} = \pm\frac{2}{3}\sqrt{2}$. Since $f''(-\frac{2}{3}\sqrt{2}) < 0$, $-\frac{2}{3}\sqrt{2}$ is a local maximum, and since $f''(\frac{2}{3}\sqrt{2}) > 0$, $\frac{2}{3}\sqrt{2}$ is a local minimum. This information is sketched in Fig. 3.3.5. ▲

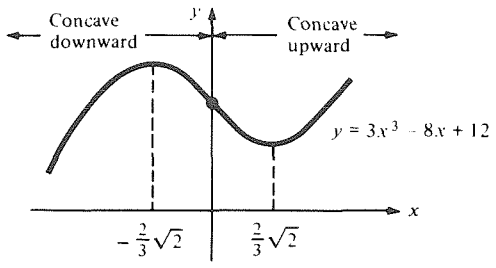


**Figure 3.3.5.** The critical points and concavity of $3x^3 - 8x + 12$.

We have just seen that a function $f$ is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Points which separate intervals where $f$ has the two types of concavity are of special interest and are called *inflection points*. More formally, we say that the point $x_0$ is a *inflection point* for the function $f$ if $f$ is twice differentiable near $x_0$ and $f''$ changes sign at $x_0$. (See Fig. 3.3.6.)

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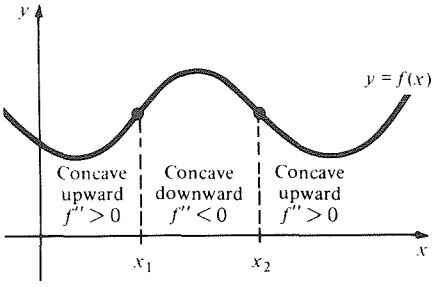


**Figure 3.3.6.** The inflection points of $f$ are $x_1$ and $x_2$.

To see where $f''$ changes sign, we begin by looking for the points where it is zero. Then we look at the next derivative, $f'''(x_0)$, to see whether a sign change actually takes place.

### Inflection Points

An inflection point for $f$ is a point where $f''$ changes sign. If $x_0$ is an inflection point for $f$, then $f''(x_0) = 0$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then $x_0$ is an inflection point for $f$.

**Example 6** Find the inflection points of $f(x) = x^2 + (1/x)$.

**Solution** The first derivative is $f'(x) = 2x - (1/x^2)$, and so the second derivative is $f''(x) = 2 + (2/x^3)$. The only possible inflection points occur where

$$0 = f''(x) = 2 + \frac{2}{x^3}.$$

That is, $x^3 = -1$; hence, $x = -1$. To test whether this is an inflection point, we calculate the third derivative: $f'''(x) = -6/x^4$, so $f'''(-1) = -6 \neq 0$; hence, $-1$ is a inflection point. ▲

If $f''(x_0) = 0$ and $f'''(x_0) = 0$, then $x_0$ may or may not be an inflection point. For example, $f(x) = x^4$ does not have an inflection point at $x_0 = 0$ (since $f''(x) = 12x^2$ does not change sign at 0), whereas $f(x) = x^5$ does have an inflection point at $x_0 = 0$ (since $f''(x) = 20x^3$ does change sign at 0). In both cases, $f''(x_0) = f'''(x_0) = 0$, so the test in the preceding box fails in this case.

We can also detect sign changes of $f''$ by examining the sign of $f''(x)$ in each interval between its roots.

**Example 7** Find the inflection points of the function $f(x) = 24x^4 - 32x^3 + 9x^2 + 1$.

**Solution** We have $f'(x) = 96x^3 - 96x^2 + 18x$, so $f''(x) = 288x^2 - 192x + 18$ and $f'''(x) = 576x - 192$. To find inflection points, we begin by solving $f''(x) = 0$; the quadratic formula gives $x = (4 \pm \sqrt{7})/12$. Using our knowledge of parabolas, we can conclude that $f''$ changes from positive to negative at $(4 - \sqrt{7})/12$ and from negative to positive at $(4 + \sqrt{7})/12$; thus both are inflection points. One could also evaluate $f'''((4 \pm \sqrt{7})/12)$, but this requires more computation. ▲

Some additional insight into the meaning of inflection points can be obtained by considering the motion of a moving object. If $x = f(t)$ is its position at time $t$, then the second derivative $d^2x/dt^2 = f''(t)$ is the rate of change of the velocity $dx/dt = f'(t)$ with respect to time—the *acceleration*. We assume that $dx/dt > 0$, so that the object is moving to the right on the number line. If $d^2x/dt^2 > 0$, the velocity is increasing; that is, the object is *accelerating*. If $d^2x/dt^2 < 0$, the velocity is decreasing; the object is *decelerating*. Therefore, a

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point of inflection occurs when the object switches from accelerating to decelerating or vice versa.

**Example 8** In Fig. 3.3.7, tell whether the points $x_1$ through $x_6$ are local maxima, local minima, inflection points, or none of these.

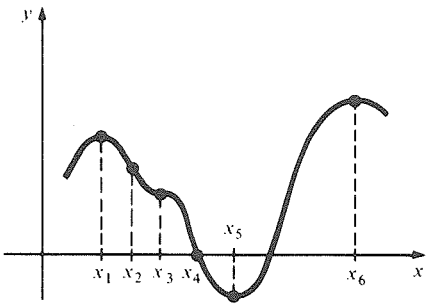


**Figure 3.3.7.** Locate the local maxima and minima and inflection points.

**Solution**

$x_1$ is a local maximum point;
$x_2$ is an inflection point;
$x_3$ is an inflection point;
$x_4$ is none of these (it is a zero of $f$);
$x_5$ is a local minimum point;
$x_6$ is a local maximum point. ▲

## Exercises for Section 3.3

Use the second derivative test to analyze the critical points of the functions in Exercises 1–8.

1. $f(x) = 3x^2 + 2$
2. $f(x) = x^3 - 6x - 3$
3. $f(x) = 6x^5 - x + 20$
4. $f(x) = x^4 - x^2$
5. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$
6. $f(x) = \dfrac{x^3 - 1}{x^2 + 1}$
7. $g(s) = \dfrac{s}{1 + s^2}$
8. $h(p) = p + \dfrac{1}{p}$

Find the intervals on which the functions in Exercises 9–16 are concave upward and those on which they are concave downward:

9. $f(x) = 3x^2 + 8x + 10$
10. $f(x) = x^3 + 3x + 8$
11. $f(x) = x^4$
12. $f(x) = \dfrac{1}{x}$
13. $f(x) = \dfrac{x}{x - 1}$
14. $f(x) = \dfrac{1}{1 + x^2}$
15. $f(x) = x^3 + 4x^2 - 8x + 1.$
16. $f(x) = (x - 2)^3 + 8.$

Find the inflection points for the functions in Exercises 17–24.

17. $f(x) = x^3 - x$
18. $f(x) = x^4 - x^2 + 1$
19. $f(x) = x^7$
20. $f(x) = x^6$
21. $f(x) = (x - 1)^4$
22. $f(x) = 2x^3 + 3x$
23. $f(x) = \dfrac{1}{1 + x^2}$
24. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

25. In each of the graphs of Fig. 3.3.8, tell whether $x_0$ is a local maximum point, a local minimum point, an inflection point, or none of these.

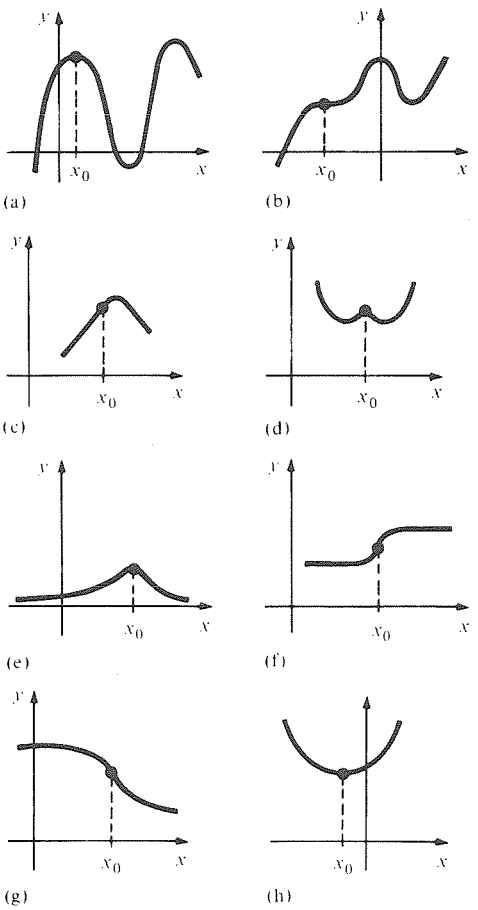


**Figure 3.3.8.** Is $x_0$ a local maximum? A local minimum? An inflection point? Neither?

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26. Identify each of the points $x_1$ through $x_6$ in Fig. 3.3.9 as a local maximum point, local minimum point, inflection point, or none of these.

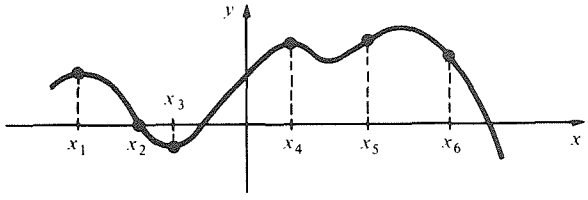


**Figure 3.3.9.** Classify the points $x_1, \ldots, x_6$.

Find the local maxima, local minima, and inflection points of each of the functions in Exercises 27–30. Also find the intervals on which each function is increasing, decreasing, and concave upward and downward.

27. $\frac{1}{4}x^2 - 1$
28. $\dfrac{1}{x(x-1)}$
29. $x^3 + 2x^2 - 4x + \frac{3}{2}$
30. $x^2 - x^4$
31. Find the inflection points for $x^n$, $n$ a positive integer. How does the answer depend upon $n$?
32. Is an inflection point always, sometimes, or never a critical point? Explain.
33. Find a function with inflection points at 1 and 2. [*Hint*: Start by writing down $f''(x)$. Then figure out what $f'(x)$ and $f(x)$ should be.]
34. Find a function with local maxima or minima at 1 and 3 and an inflection point at 2.
35. (a) Relate the sign of the error made in the linear approximation of $f$ to the second derivative of $f$. (b) Apply your conclusion to the linear approximation of $1/x$ at $x_0 = 1$.
36. (a) Use the second derivative to compare $x^2$ with $9 + 6(x - 3)$ for $x$ near 3. (b) Show by algebra that $x^2 > 9 + 6(x - 3)$ for all $x \neq 3$.
37. Let $f(x) = x^3 - x$.
    (a) Find the linear approximations to $f$ at $x_0 = -1, 0$, and 1.
    (b) For each such $x_0$, compare the value of $f(x_0 + \Delta x)$ with the linear approximation for $\Delta x = \pm 1, \pm 0.1, \pm 0.01$. How does the error depend upon $f''(x_0)$?
38. Show that if $f'(x_0) = f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then $x_0$ is *not* a local maximum or local minimum point for $f$.
39. The power output of a battery is given by $P = EI - RI^2$, where $E$ and $R$ are positive constants.
    (a) For which current $I$ is the power $P$ a local maximum? Justify using the second derivative test.
    (b) What is the maximum power?
40. A generator of $E$ volts is connected to an inductor of $L$ henrys, a resistor of $R$ ohms, and a second resistor of $x$ ohms. Heat is dissipated from the second resistor, the power $P$ being given by
$$P = \frac{E^2 x}{(2\pi L)^2 + (x + R)^2}.$$
    (a) Find the resistance value $x_0$ which makes the power as large as possible. Justify with the second derivative test.
    (b) Find the maximum power which can be achieved by adjustment of the resistance $x$.
41. A rock thrown upward attains a height $s = 3 + 40t - 16t^2$ feet in $t$ seconds. Using the second derivative test, find the maximum height of the rock.
42. An Idaho cattle rancher owns 1600 acres adjacent to the Snake River. He wishes to make a three-sided fence from 2 miles of surplus fencing, the enclosure being set against the river to make a rectangular corral. If $x$ is the length of the short side of the fence, then $A = x(2 - 2x)$ is the area enclosed by the fence (assuming the river is straight).
    (a) Show that the maximum area occurs when $x = \frac{1}{2}$, using the second derivative test.
    (b) Verify that the maximum area enclosed is 0.5 square mile.
    (c) Verify that the fence dimensions are $\frac{1}{2}$, 1, $\frac{1}{2}$ miles, when the area enclosed is a maximum.

★43. Sketch the graphs of continuous functions on $(-\infty, \infty)$ with the following descriptions. (If you think no such function can exist, state that as your answer.)
    (a) Three local maxima or minima and two points of inflection.
    (b) Two local maxima or minima and three points of inflection.
    (c) Four local maxima or minima and no points of inflection.
    (d) Two (strict) local maxima and no (strict) local minima.

★44. Suppose that $f'(x_0) = f''(x_0) = f'''(x_0) = 0$, but $f''''(x_0) \neq 0$. Is $x_0$ a local maximum point, a local minimum point, or an inflection point of $f$? Give examples to show that anything can happen if $f''''(x_0) = 0$.

★45. If $f(x)$ is positive for all $x$, do $f(x)$ and $1/f(x)$ have the same inflection points?

★46. Prove that no odd-degree polynomial can be everywhere concave upward. (As part of your solution, give a few simple examples and include a brief discussion of the possibilities for even-degree polynomials.)

★47. Prove the following theorem, which shows that the tangent line at a point of inflection crosses the graph:

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*Let $x_0$ be an inflection point for $f$ and let $h(x) = f(x) - [f(x_0) + f'(x_0)(x - x_0)]$ be the difference between $f$ and its linear approximation at $x_0$. If $f''$ changes sign from negative to positive [or positive to negative] at $x_0$ (for example, if $f'''(x_0) > 0$ [or $< 0$]), then $h$ changes sign from negative to positive [or positive to negative] at $x_0$.*

The two cases are illustrated in Fig. 3.3.10.

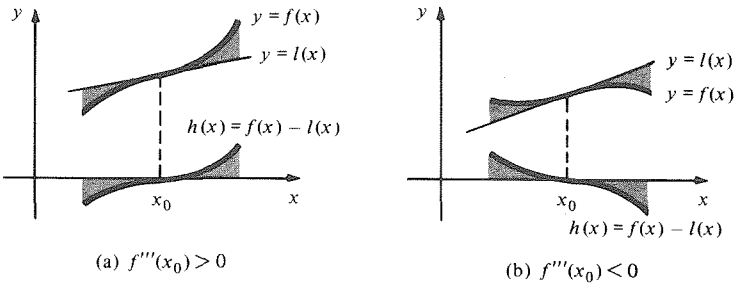


**Figure 3.3.10.** The graph of $f$ crosses its tangent line at a point of inflection.

# 3.4 Drawing Graphs

*Using calculus to determine the principal features of a graph often produces better results than simple plotting.*

One of the best ways to understand a function is to see its graph. The simplest way to draw a graph is by plotting some points and connecting them with a smooth curve, but this method can lead to serious errors unless we are sure that we have plotted enough points. The methods described in the first three sections of this chapter, combined with the techniques of differentiation, help us make a good choice of which points to plot and show us how to connect the points by a curve of the proper shape.

We begin by outlining a systematic procedure to follow in graphing any function.

## Graphing Procedure

To sketch the graph of a function $f$:

1. Note any *symmetries* of $f$. Is $f(x) = f(-x)$, or $f(x) = -f(-x)$, or neither? In the first case, $f$ is called *even*; in the second case, $f$ is called *odd*. (See Fig. 3.4.1 and the remarks below.)
2. Locate any points where $f$ is not defined and determine the behavior of $f$ near these points. Also determine, if you can, the behavior of $f(x)$ for $x$ very large positive and negative.
3. Locate the local maxima and minima of $f$, and determine the intervals on which $f$ is increasing and decreasing.
4. Locate the inflection points of $f$, and determine the intervals on which $f$ is concave upward and downward.
5. Plot a few other key points, such as $x$ and $y$ intercepts, and draw a small piece of the tangent line to the graph at each of the points you have plotted. (To do this, you must evaluate $f'$ at each point.)
6. Fill in the graph consistent with the information gathered in steps 1 through 5.

Let us examine the graphing procedure beginning with step 1. If $f$ is even—that is, $f(x) = f(-x)$—we may plot the graph for $x \geqslant 0$ and then reflect the

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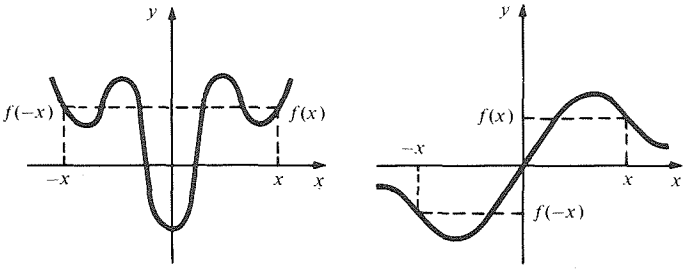
result across the $y$ axis to obtain the graph for $x \leqslant 0$. (See Fig. 3.4.1 (a).) If $f$ is odd, that is, $f(x) = -f(-x)$ then, having plotted $f$ for $x \geqslant 0$, we may reflect the graph in the $y$ axis and then in the $x$ axis to obtain the graph for $x \leqslant 0$. (See Fig. 3.4.1 (b).)




(a) Even function (b) Odd function

**Figure 3.4.1.** $f$ is even when $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$.

To decide whether a function is even or odd, substitute $-x$ for $x$ in the expression for $f(x)$ and see if the resulting expression is the same as $f(x)$, the negative of $f(x)$, or neither.

**Example 1** Classify each of the following functions as even, odd, or neither:

(a) $f(x) = x^4 + 3x^2 + 12$;
(b) $g(x) = x/(1 + x^2)$;
(c) $h(x) = x/(1 + x)$;

**Solution** (a) $f(-x) = (-x)^4 + 3(-x)^2 + 12 = x^4 + 3x^2 + 12 = f(x)$, so $f$ is even.
(b) $g(-x) = (-x)/(1 + (-x)^2) = -x/(1 + x^2) = -g(x)$, so $g$ is odd.
(c) $h(-x) = (-x)/(1 + (-x)) = -x/(1 - x)$, which does not appear to equal $h(x)$ or $-h(x)$. To be sure, we substitute $x = 2$, for which $h(x) = \frac{2}{3}$ and $h(-x) = 2$; thus, $h$ is neither even nor odd. ▲

Step 2 is concerned with what is known as the *asymptotic* behavior of the function $f$ and is best explained through an example. The asymptotic behavior involves infinite limits of the type $\lim_{x \to x_0} f(x) = \pm\infty$ and $\lim_{x \to \pm\infty} f(x) = l$, as were discussed in Section 1.2.

**Example 2** Find the asymptotic behavior of $f(x) = x/(1 - x)$ ($f$ is not defined for $x = 1$).

**Solution** For $x$ near 1 and $x > 1$, $1 - x$ is a small negative number, so $f(x) = x/(1 - x)$ is large and negative; for $x$ near 1 and $x < 1$, $1 - x$ is small and positive, so $x/(1 - x)$ is large and positive. Thus we could sketch the part of the graph of $f$ near $x = 1$ as in Fig. 3.4.2. The line $x = 1$ is called a *vertical asymptote* for $x/(1 - x)$. In terms of limits, we write $\lim_{x \to 1-}[x/(1 - x)] = +\infty$ and $\lim_{x \to 1+}[x/(1 - x)] = -\infty$.

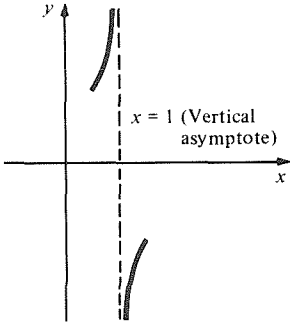



**Figure 3.4.2.** Pieces of the graph $y = x/(1 - x)$ are plotted near the vertical asymptote $x = 1$.

Next we examine the behavior of $x/(1 - x)$ when $x$ is large and positive and when $x$ is large and negative. Since both the numerator and denominator also become large, it is not clear what the ratio does. We may note, however, that

$$\frac{x}{1 - x} = \frac{1}{(1/x) - 1}$$

for $x \neq 0$. As $x$ becomes large (positive or negative), $1/x$ becomes small, and $(1/x) - 1$ approaches $-1$, so $1/[(1/x) - 1]$ approaches $1/(-1) = -1$, i.e., $\lim_{x \to \infty}[x/(1 - x)] = -1$ and $\lim_{x \to -\infty}[x/(1 - x)] = -1$. Furthermore, when $x$ is large and positive, $1/x > 0$, so $(1/x) - 1 > -1$, and therefore $1/[(1/x) - 1] < 1/(-1) = -1$, so the graph lies near, but below, the line

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$y = -1$. Similarly, for $x$ large and negative, $1/x < 0$, so $(1/x) - 1 < -1$, and therefore $1/[(1/x) - 1] > -1$, so the graph lies near, but above, the line $y = -1$. Thus, we could sketch the part of the graph for $x$ large as in Fig. 3.4.3. The line $y = -1$ is called a *horizontal asymptote* for $f$. ▲

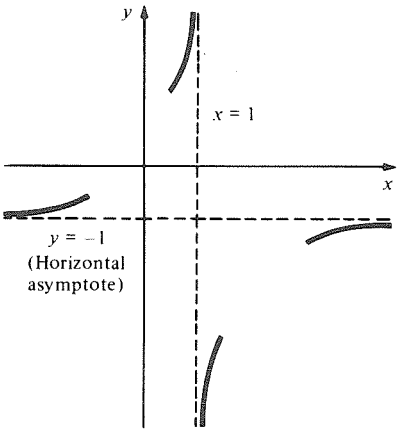


**Figure 3.4.3.** Pieces of the graph $y = x/(1 - x)$ are plotted near the horizontal asymptote $y = -1$.

Steps 3 and 4 were described in detail in Sections 3.2 and 3.3; step 5 increases the accuracy of plotting, and step 6 completes the job. These steps will be carried out in detail in the examples that follow. The graph $y = x/(1 - x)$ begun above is discussed again in Example 9.

Some words of advice: It is important to be systematic; follow the procedure step by step, and introduce the information on the graph as you proceed. A haphazard attack on a graph often leads to confusion and sometimes to desperation. Just knowing steps 1 through 6 is not enough—you must be able to employ them effectively. The only way to develop this ability is through practice.

**Example 3** Sketch the graph of $f(x) = x - \dfrac{1}{x}$.

**Solution** We carry out the six-step procedure:

1. $f(-x) = -x + (1/x) = -f(x)$; $f$ is odd, so we need only study $f(x)$ for $x \geqslant 0$.
2. $f$ is not defined for $x = 0$. For $x$ small and positive, $-(1/x)$ is large in magnitude and negative in sign, so $x - (1/x)$ is large and negative as well; $x = 0$ is a vertical asymptote. For $x$ large and positive, $-(1/x)$ is small and negative; thus the graph of $f(x) = x - (1/x)$ lies below the line $y = x$, approaching the line as $x$ becomes larger. The line $y = x$ is again called an *asymptote* (see Fig. 3.4.4).

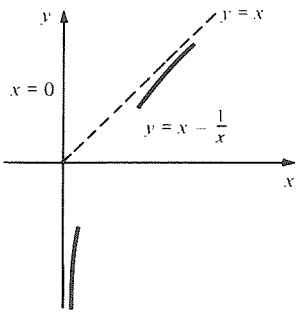


**Figure 3.4.4.** The lines $x = 0$ and $y = x$ are asymptotes.

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3. $f'(x) = 1 + (1/x^2)$, which is positive for all $x \neq 0$. Thus $f$ is always increasing and there are no maxima or minima.
4. $f''(x) = -(2/x^3)$, which is negative for all $x > 0$; $f$ is concave downward on $(0, \infty)$.
5. The intercept occurs where $x - (1/x) = 0$; that is, $x = 1$. We have $f'(1) = 2$, $f(2) = \frac{3}{2}$, $f'(2) = \frac{5}{4}$.

The information obtained in steps 1 through 5 is placed on the graph in Fig. 3.4.5.

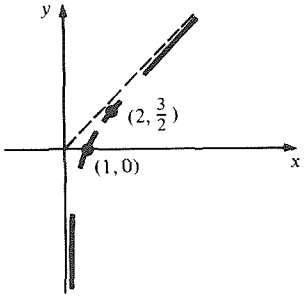


**Figure 3.4.5.** The information obtained from steps 1 to 5.

6. We fill in the graph for $x > 0$ (Fig. 3.4.6). Finally, we use the fact that $f$ is odd to obtain the other half of the graph by reflecting through the $x$ and $y$ axes. (See Fig. 3.4.7.) ▲

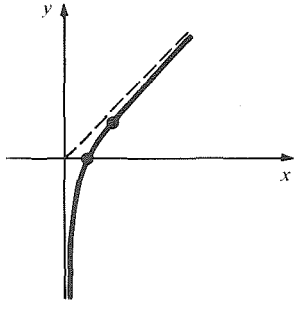


**Figure 3.4.6.** The graph for $x > 0$ is filled in (step 6).

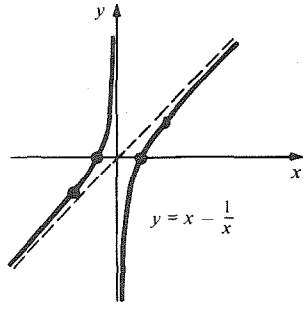


**Figure 3.4.7.** The complete graph is obtained by using the fact that $f$ is odd.

## Calculator Remark

While calculators enable one to plot points relatively quickly, and computers will plot graphs from formulas, the use of calculus is still essential. A calculator can be deceptive if used alone, as we saw in Example 7, Section R.6. In Chapter 14 we will see how the computer can help us graph complicated surfaces in space, but it may be unwise to begin expensive computation before a thorough analysis using calculus. Of course, it may be even quicker to solve a simple problem by calculus than to go to a machine for plotting. ▲

**Example 4** Sketch the graph of $f(x) = \dfrac{x}{1 + x^2}$.

**Solution** Again we carry out the six-step procedure:

1. $f(-x) = -x/[1 + (-x)^2] = -x/[1 + x^2] = -f(x)$; $f$ is odd, so its graph must by symmetric when reflected in the $x$ and $y$ axes.
2. Since the denominator $1 + x^2$ is never zero, the function is defined everywhere; there are no vertical asymptotes. For $x \neq 0$, we have

$$f(x) = \frac{x}{1 + x^2} = \frac{1}{x + (1/x)}.$$

Since $1/x$ becomes small as $x$ becomes large, $f(x)$ looks like $1/(x + 0) = 1/x$ for $x$ large. Thus $y = 0$ is a horizontal asymptote; the graph is below $y = 0$ for $x$ large and negative and above $y = 0$ for $x$ large and positive.

3. $$f'(x) = \frac{1 + x^2 - x(2x)}{(1 + x^2)^2} = \frac{(1 - x^2)}{(1 + x^2)^2},$$

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which vanishes when $x = \pm 1$. To check the sign of $f'(x)$ on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, we evaluate it at conveniently chosen points: $f'(-2) = -\frac{3}{25}$, $f'(0) = 1$, $f'(2) = -\frac{3}{25}$. Thus $f$ is decreasing on $(-\infty, -1)$ and on $(1, \infty)$ and $f$ is increasing on $(-1, 1)$. Hence $-1$ is a local minimum and 1 is a local maximum by the first derivative test.

4. $$f''(x) = \frac{(1+x^2)^2(-2x) - (1-x^2)\cdot 2(1+x^2)2x}{(1+x^2)^4} = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

This is zero when $x = 0$, $\sqrt{3}$, and $-\sqrt{3}$. Since the denominator of $f''$ is positive, we can determine the sign by evaluating the numerator. Evaluating at $-2$, $-1$, 1, and 2, we get $-4$, 4, $-4$, and 4, so $f$ is concave downward on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ and concave upward on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$; $-\sqrt{3}$, 0, and $\sqrt{3}$ are points of inflection.

5. $$f(0) = 0; \qquad f'(0) = 1,$$
$$f(1) = \tfrac{1}{2}; \qquad f'(1) = 0,$$
$$f(\sqrt{3}) = \tfrac{1}{4}\sqrt{3}; \qquad f'(\sqrt{3}) = -\tfrac{1}{8}.$$

The only solution of $f(x) = 0$ is $x = 0$.

The information obtained in steps 1 through 5 is placed tentatively on the graph in Fig. 3.4.8. As we said in step 1, we need do this only for $x \geqslant 0$.

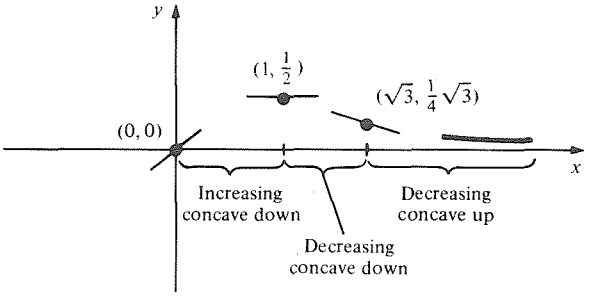


**Figure 3.4.8.** The graph $y = x/(1 + x^2)$ after steps 1 to 5.

6. We draw the final graph, remembering to obtain the left-hand side by reflecting the right-hand side in both axes. (You can get the same effect by rotating the graph 180°, keeping the origin fixed.) The result is shown in Fig. 3.4.9. ▲

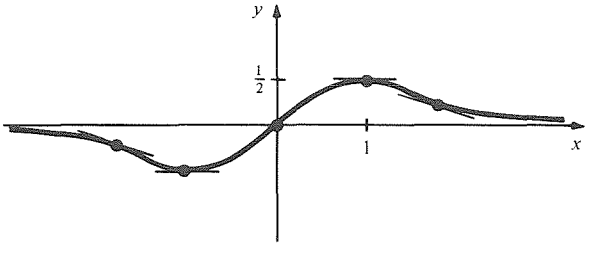


**Figure 3.4.9.** The complete graph $y = x/(1 + x^2)$.

**Example 5** Sketch the graph of $f(x) = 2x^3 + 8x + 1$.

**Solution** The six steps are as follows:

1. $f(-x) = -2x^3 - 8x + 1$ is not equal to $f(x)$ or $-f(x)$, so $f$ is neither even nor odd.

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2. $f$ is defined everywhere. We may write

$$f(x) = 2x^3\left(1 + \frac{4}{x^2} + \frac{1}{2x^3}\right).$$

For $x$ large, the factor $1 + (4/x^2) + (1/2x^3)$ is near 1, so $f(x)$ is large and positive for $x$ large and positive, and large and negative for $x$ large and negative. There are no horizontal or vertical asymptotes.

3. $f'(x) = 6x^2 + 8$, which vanishes nowhere and is always positive. Thus $f$ is increasing on $(-\infty, \infty)$ and has no critical points.

4. $f''(x) = 12x$, which is negative for $x < 0$ and positive for $x > 0$. Thus $f$ is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$; zero is a point of inflection.

5. $f(0) = 1; \quad f'(0) = 8,$
$f(1) = 11; \quad f'(1) = 14,$
$f(-1) = -9; \quad f'(-1) = 14.$

The information obtained so far is plotted in Fig. 3.4.10.

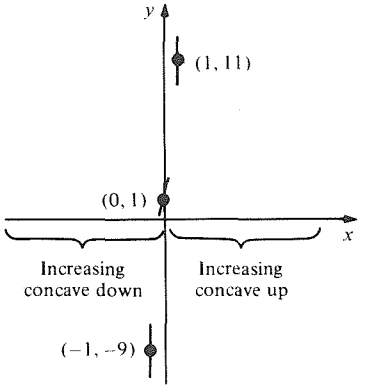


**Figure 3.4.10.** The graph $y = 2x^3 + 8x + 1$ after steps 1 to 5.

6. A look at Fig. 3.4.10 suggests that the graph will be very long and thin. In fact, $f(2) = 33$, which is way off the graph. To get a useful picture, we may stretch the graph horizontally by changing units on the $x$ axis so that a unit on the $x$ axis is, say, four times as large as a unit on the $y$ axis. We add a couple of additional points by calculating

$$f(\tfrac{1}{2}) = 5\tfrac{1}{4}; \qquad f'(\tfrac{1}{2}) = 9\tfrac{1}{2},$$
$$f(-\tfrac{1}{2}) = -3\tfrac{1}{4}; \qquad f'(-\tfrac{1}{2}) = 9\tfrac{1}{2}.$$

Then we draw a smooth curve as in Fig. 3.4.11. ▲

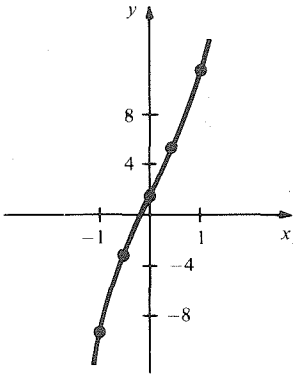


**Figure 3.4.11.** The completed graph $y = 2x^3 + 8x + 1$.

Any cubic function $y = ax^3 + bx^2 + cx + d$ may be plotted just as the one in the preceding example. The critical points are obtained by solving the quadratic equation

$$f'(x) = 3ax^2 + 2bx + c = 0$$

which may have one or two roots or none. In Example 5, $f'(x) = 0$ had no roots, and $f'$ was always positive. If $f'$ has two roots, $y = f(x)$ will have one local maximum and one local minimum. Let us do an example of this type.

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**Example 6** Sketch the graph of $f(x) = 2x^3 - 8x + 1$.

**Solution** Again, we use the six-step procedure:

1. There are no symmetries.
2. There are no asymptotes. As in Example 5, $f(x)$ is large positive for $x$ large positive and large negative for $x$ large negative.
3. $f'(x) = 6x^2 - 8$, which is zero when $x = \pm\sqrt{4/3} = \pm 2/\sqrt{3} \approx \pm 1.15$. Also, $f'(-2) = f'(2) = 16$ and $f'(0) = -8$, so $f$ is increasing on $(-\infty, -2/\sqrt{3}\,]$ and $[2/\sqrt{3}\,, \infty)$ and decreasing on $[-2/\sqrt{3}\,, 2/\sqrt{3}\,]$. Thus, $-2/\sqrt{3}$ is a local maximum point and $2/\sqrt{3}$ a local minimum point.
4. $f''(x) = 12x$, so $f$ is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$. Zero is an inflection point.
5. $f(0) = 1$; $f'(0) = -8$,
   $f(-2/\sqrt{3}\,) = 1 + 32/3\sqrt{3} \approx 7.16$; $f'(-2/\sqrt{3}\,) = 0$,
   $f(2/\sqrt{3}\,) = 1 - 32/3\sqrt{3} \approx -5.16$; $f'(2/\sqrt{3}\,) = 0$,
   $f(-\frac{1}{2}) = 4\frac{3}{4}$; $f'(-\frac{1}{2}) = -6\frac{1}{2}$,
   $f(\frac{1}{2}) = -2\frac{3}{4}$; $f'(\frac{1}{2}) = -6\frac{1}{2}$,
   $f(-2) = 1$; $f'(-2) = 16$,
   $f(2) = 1$; $f'(2) = 16$.

   The data are plotted in Fig. 3.4.12. The scale is stretched by a factor of 4 in the $x$ direction, as in Fig. 3.4.11.
6. We draw the graph (Fig. 3.4.13). ▲

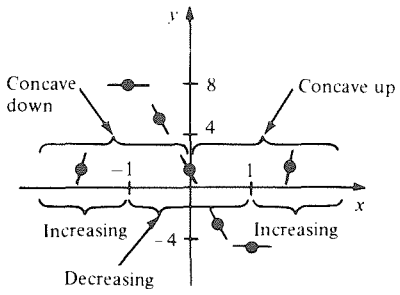


**Figure 3.4.12.** The graph $y = 2x^3 - 8x + 1$ after steps 1 to 5.

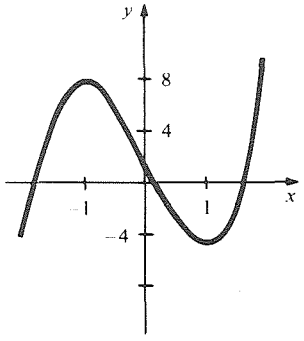


**Figure 3.4.13.** The completed graph $y = 2x^3 - 8x + 1$.

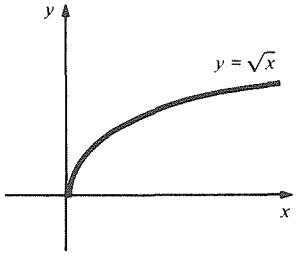


**Figure 3.4.14.** The graph $y = \sqrt{x}$.

Some interesting new features arise when we graph functions which involve fractional powers. For example, consider the graph of $y = \sqrt{x}$ for $x \geqslant 0$. Notice that the slope $dy/dx = \frac{1}{2}x^{-1/2} = 1/(2\sqrt{x}\,)$ becomes large and positive as $x \to 0$, while $dy/dx$ approaches 0 as $x \to \infty$. Thus, the graph appears as in Fig. 3.4.14.

Something similar happens for the cube root $y = x^{1/3}$, so that $dy/dx = 1/(3x^{2/3})$. This time, the function is defined for all $x$; its derivative exists for $x \neq 0$ and is large and positive for $x$ near 0, of both signs. (See Fig. 3.4.15.) We sometimes say that the graph $y = x^{1/3}$ has a *vertical tangent* at $x = 0$.

You may notice that the graphs $y = x^{1/2}$ and $y = x^{1/3}$ resemble $y = x^2$

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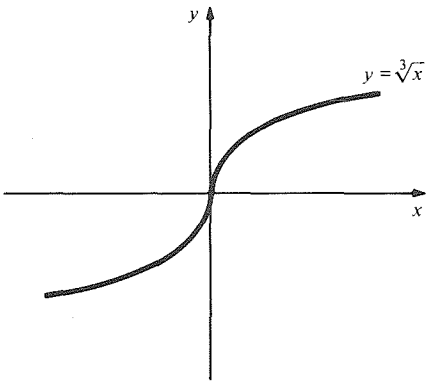


**Figure 3.4.15.** The graph $y = \sqrt[3]{x}$ has a "vertical tangent line" at the origin.

and $y = x^3$ turned on their sides. This relationship will be explored when we study inverse functions in Section 5.3.

Still more interesting is the graph of $y = x^{2/3}$, which is also defined for all $x$. The derivative is $dy/dx = \frac{2}{3}x^{-1/3} = 2/(3\sqrt[3]{x})$. For $x$ near 0 and positive, $dy/dx$ is large and positive, whereas for $x$ near 0 and negative, $dy/dx$ is large and negative. Thus, the graph has the appearance shown in Fig. 3.4.16. Again,

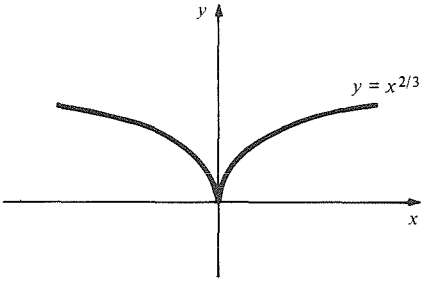


**Figure 3.4.16.** The graph $y = x^{2/3}$ has a cusp at the origin.

we can say that the graph has a vertical tangent at $x = 0$. However, the shape of the graph near $x = 0$ has not been encountered before. We call $x = 0$ a *cusp*. Note that $x = 0$ is a minimum point of $f(x) = x^{2/3}$, but that $x^{2/3}$ is not differentiable there.

In general, a continuous function $f$ is said to have a *cusp* at $x_0$ if $f'(x)$ has opposite signs on opposite sides of $x_0$ but $f'(x)$ "blows up" at $x_0$ in the sense that $\lim_{x\to x_0}[1/f'(x)] = 0$; thus $\lim_{x\to x_0\pm} f'(x) = \pm\infty$ or $\mp\infty$.

**Example 7** Let $f(x) = (x^2 + 1)^{3/2}$. (a) Where is $f$ increasing? (b) Sketch the graph of $f$. Are there any cusps?

**Solution** (a) Using the chain rule, we get $f'(x) = 3x\sqrt{x^2 + 1}$. Hence $f'(x) < 0$ (so $f$ is decreasing) on $(-\infty, 0)$, and $f'(x) > 0$ (so $f$ is increasing) on $(0, \infty)$.

(b) By the first derivative test, $x = 0$ is a local minimum point. Note that $f(x)$ is an even function and

$$f''(x) = 3\left(\frac{x^2}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1}\right) > 0,$$

so $f$ is concave upward. Thus we can sketch its graph as in Fig. 3.4.17. There are no cusps. ▲

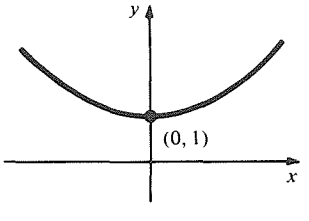


**Figure 3.4.17.** The graph $y = (x^2 + 1)^{3/2}$.

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**Example 8** Sketch the graph of $(x+1)^{2/3}x^2$.

**Solution** Letting $f(x) = (x+1)^{2/3}x^2$, we have

$$f'(x) = \tfrac{2}{3}(x+1)^{-1/3}x^2 + (x+1)^{2/3}\cdot 2x$$
$$= \left[2x/3(x+1)^{1/3}\right](4x+3).$$

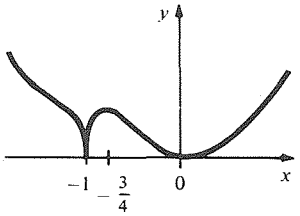


**Figure 3.4.18.** The graph $y = x^2(x+1)^{2/3}$ has a cusp at $x = -1$.

For $x$ near $-1$, but $x > -1$, $f'(x)$ is large positive, while for $x < -1$, $f'(x)$ is large negative. Since $f$ is continuous at $-1$, this is a local minimum and a cusp.

The other critical points are $x = 0$ and $x = -\frac{3}{4}$. From the first derivative test (or second derivative test, if you prefer), $-\frac{3}{4}$ is a local maximum and zero is a local minimum. For $x > 0$, $f$ is increasing since $f'(x) > 0$; for $x < -1$, $f$ is decreasing since $f'(x) < 0$. Thus we can sketch the graph as in Fig. 3.4.18. (We located the inflection points at $(-33 \pm \sqrt{609})/40 \approx -0.208$ and $-1.442$ by setting the second derivative equal to zero.) ▲

Sometimes algebraic transformations simplify the job of drawing a graph.

**Example 9** Sketch the graph $y = x/(1-x)$ by (a) the six-step procedure and (b) by making the transformation $u = 1 - x$.

**Solution** (a) In Example 1 we carried out steps 1 and 2. To carry out step 3, we compute using the quotient rule:

$$\frac{dy}{dx} = \frac{(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

Since $dy/dx$ is always positive (undefined if $x = 1$), the graph has no maxima and minima and the function is increasing on the intervals $(-\infty, 1)$ and $(1, \infty)$. For step 4, we compute that

$$\frac{d^2y}{dx^2} = \frac{2}{(1-x)^3},$$

so the graph is concave upward on $(-\infty, 1)$ and concave downward on $(1, \infty)$. For step 5 we note that $y = 0$ when $x = 0$ and $y = -2$ when $x = 2$. At both of these points $dy/dx$ is 1. The graph is then plotted in Fig. 3.4.19.

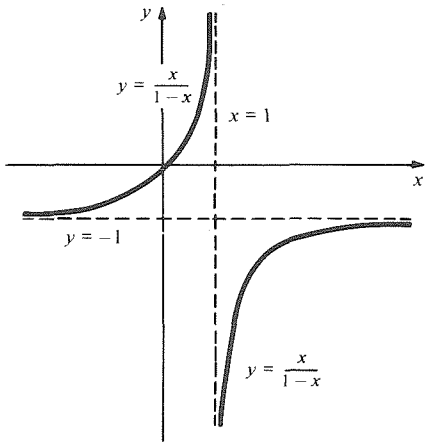


**Figure 3.4.19.** The graph $y = x/(1-x)$.

(b) If we let $u = x - 1$, then $x = 1 + u$, so $y = (1+u)/(-u) = -1 - 1/u$. Letting $z = y + 1$, we get $z = -1/u$. This graph is easy to sketch and may already be familiar to you as a hyperbola in the second and fourth

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quadrants of the $uz$ plane. Translating axes (just as we did for parabolas in Section R.2) produces the same result as in Fig. 3.4.19, in which the dashed lines are the $u$ and $z$ axes. ▲

## Supplement to Section 3.4 The General Cubic

In Section R.2, we saw how to plot the general linear function, $f(x) = ax + b$, and the general quadratic function, $f(x) = ax^2 + bx + c$ ($a \neq 0$). The methods of this section yield the same results: the graph of $ax + b$ is a straight line, while the graph of $f(x) = ax^2 + bx + c$ is a parabola, concave upward if $a > 0$ and concave downward if $a < 0$. Moreover, the maximum (or minimum) point of the parabola occurs when $f'(x) = 2ax + b = 0$, that is, $x = -(b/2a)$, which is the same result as was obtained by completing the square.

A more ambitious task is to determine the shape of the graph of the general cubic $f(x) = ax^3 + bx^2 + cx + d$. (We assume that $a \neq 0$; otherwise, we are dealing with a quadratic or linear function.) Of course, any specific cubic can be plotted by techniques already developed, but we wish to get an idea of what *all possible* cubics look like and how their shapes depend on $a$, $b$, $c$, and $d$.

We begin our analysis with some simplifying transformations. First of all, we can factor out $a$ and obtain a new polynomial $f_1(x)$ as follows:

$$f(x) = a\left(x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right) = af_1(x).$$

The graphs of $f$ and $f_1$ have the same basic shape; if $a > 0$, the $y$ axis is just rescaled by multiplying all $y$ values by $a$; if $a < 0$, the $y$ axis is rescaled and the graph is flipped about the $x$ axis. It follows that we not lose any generality by assuming that the coefficient of $x^3$ is 1.

Therefore, consider the simpler form

$$f_1(x) = x^3 + b_1x^2 + c_1x + d_1,$$

where $b_1 = b/a$, $c_1 = c/a$, and $d_1 = d/a$. In trying to solve cubic equations, mathematicians of the early Renaissance noticed a useful trick: if we replace $x$ by $x - (b_1/3)$, then the quadratic term drops out; that is,[3]

$$f_1\left(x - \frac{b_1}{3}\right) = x^3 + c_2x + d_2,$$

[3] These algebraic ideas are related to a formula for the roots of a general cubic, discovered by Niccolo Tartaglia (1506–1559) but published by (without Tartaglia's permission) and often credited to Girolamo Cardano (1501–1576). Namely, the solutions of the cubic equation $x^3 + bx^2 + cx + d = 0$ are

$$x_1 = S + T - \frac{b}{3}, \qquad \text{where} \quad S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}$$

$$x_2 = -\frac{1}{2}(S + T) - \frac{b}{3} + \frac{1}{2}\sqrt{-3}\,(S - T), \qquad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

$$x_3 = -\frac{1}{2}(S + T) - \frac{b}{3} - \frac{1}{2}\sqrt{-3}\,(S - T), \quad Q = \frac{3c - b^2}{9}, \quad R = \frac{9bc - 27d - 2b^3}{54}$$

There is also a formula for the roots of a quartic equation, but a famous theorem (due to Abel and Ruffini in the nineteenth century) states that there can be no such algebraic formula for the general equation of degree $\geq 5$. Modern proofs of this theorem can be found in advanced textbooks on the algebra [such as L. Goldstein, *Abstract Algebra*, Prentice-Hall, (1973).] These proofs are closely related to the proof of the impossibility of trisecting angles with ruler and compass.

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where $c_2$ and $d_2$ are new constants, depending on $b_1$, $c_1$, and $d_1$. (We leave to the reader the task of verifying this last statement and expressing $c_2$ and $d_2$ in terms of $b_1$, $c_1$, and $d_1$; see Exercise 56.)

The graph of $f_1(x - b_1/3) = f_2(x)$ is the same as that of $f_1(x)$ except that it is shifted by $b_1/3$ units along the $x$ axis. This means that we lose no generality by assuming that the coefficient of $x^2$ is zero—that is, we only need to graph $f_2(x) = x^3 + c_2x + d_2$. Finally, replacing $f_2(x)$ by $f_3(x) = f_2(x) - d_2$ just corresponds to shifting the graph $d_2$ units along the $y$ axis.

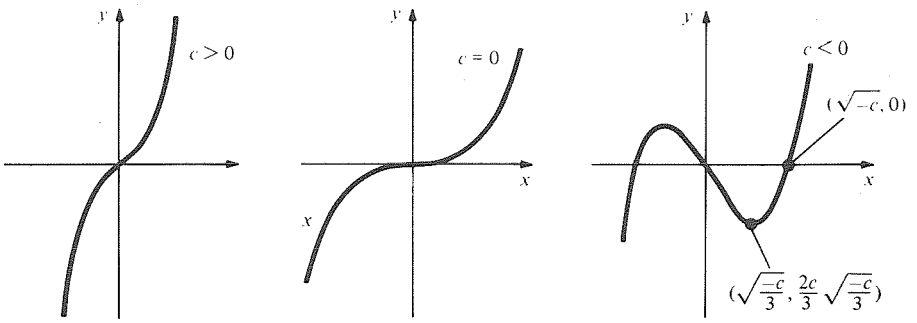
We have now reduced the graphing of the general cubic to the case of graphing $f_3(x) = x^3 + c_2x$. For simplicity let us write $f(x)$ for $f_3(x)$ and $c$ for $c_2$. To plot $f(x) = x^3 + cx$, we go through steps 1 to 6:

1. $f$ is odd.
2. $f$ is defined everywhere. Since $f(x) = x^3(1 + c/x^2)$, $f(x)$ is large and positive (negative) when $x$ is large and positive (negative); there are no horizontal or vertical asymptotes.
3. $f'(x) = 3x^2 + c$. If $c > 0$, $f'(x) > 0$ for all $x$, and $f$ is increasing everywhere. If $c = 0$, $f'(x) > 0$ except at $x = 0$, so $f$ is increasing everywhere even though the graph has a horizontal tangent at $x = 0$. If $c < 0$, $f'(x)$ has roots at $\pm\sqrt{-c/3}$; $f$ is increasing on $(-\infty, -\sqrt{-c/3}\,]$ and $[\sqrt{-c/3}\,, \infty)$ and decreasing on $[-\sqrt{-c/3}\,, \sqrt{-c/3}\,]$. Thus, $-\sqrt{-c/3}$ is a local maximum point and $\sqrt{-c/3}$ is a local minimum point.
4. $f''(x) = 6x$, so $f$ is concave downward for $x < 0$ and concave upward for $x > 0$. Zero is an inflection point.
5. $f(0) = 0$, $f'(0) = c$,

$$f(\pm\sqrt{-c}) = 0, \qquad f'(\pm\sqrt{-c}) = -2c \quad (\text{if} \quad c < 0);$$

$$f\left(\pm\sqrt{\frac{-c}{3}}\right) = \pm\frac{2}{3}c\sqrt{\frac{-c}{3}}, \qquad f'\left(\pm\sqrt{\frac{-c}{3}}\right) = 0 \quad (\text{if} \quad c < 0).$$

6. We skip the preliminary sketch and draw the final graphs. (See Fig. 3.4.20.)




(a) Type I (b) Type II (c) Type III

**Figure 3.4.20.** The graph of $y = x^3 + cx$ for: $c > 0$ (I); $c = 0$ (II); and $c < 0$ (III).

Thus there are three types of cubics (with $a > 0$):

Type I: $f$ is increasing at all points, $f' > 0$ everywhere.
Type II: $f$ is increasing at all points; $f' = 0$ at one point.
Type III: $f$ has a maximum and a minimum.

Type II is the transition type between types I and III. You may imagine the graph changing as $c$ begins with a negative value and then moves toward zero. As $c$ gets smaller and smaller, the turning points move in toward the

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origin and the bumps merge at the point where $f'(x) = 0$. As $c$ passes zero to become positive, the bumps disappear completely.

**Example 10** Convert $2x^3 + 3x^2 + x + 1$ to the form $x^3 + cx$ and determine whether the cubic is of type I, II, or III.

**Solution** $\frac{1}{2}(2x^3 + 3x^2 + x + 1) = x^3 + \frac{3}{2}x^2 + \frac{1}{2}x + \frac{1}{2}$. Substituting $x - \frac{1}{2}$ for $x$ (since the coefficient of $x^2$ is $\frac{3}{2}$) gives

$$\begin{aligned}&(x - \tfrac{1}{2})^3 + \tfrac{3}{2}(x - \tfrac{1}{2})^2 + \tfrac{1}{2}(x - \tfrac{1}{2}) + \tfrac{1}{2}\\ &\quad = x^3 - \tfrac{3}{2}x^2 + \tfrac{3}{4}x - \frac{1}{8} + \tfrac{3}{2}(x^2 - x + \tfrac{1}{4}) + \tfrac{1}{2}(x - \tfrac{1}{2}) + \tfrac{1}{2}\\ &\quad = x^3 - \tfrac{1}{4}x + \tfrac{1}{2}.\end{aligned}$$

Thus, after being shifted along the $x$ and $y$ axes, the cubic becomes $x^3 - \frac{1}{4}x$. Since $c < 0$, it is of type III. ▲

## Exercises for Section 3.4

In Exercises 1–4, is the function even, odd, or neither?

1. $f(x) = \dfrac{x^3 + 6x}{x^2 + 1}$
2. $f(x) = x$
3. $f(x) = \dfrac{x}{x^3 + 1}$
4. $f(x) = x^6 + 8x^2 + 3$

In Exercises 5–8, describe the behavior of the given functions near their vertical asymptotes:

5. $f(x) = \dfrac{x}{x^3 + 1}$
6. $f(x) = \dfrac{x}{x^3 - 1}$
7. $f(x) = \dfrac{x^3 + 1}{x^2 - 1}$
8. $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$

Sketch the graphs of the functions in Exercises 9–30.

9. $\dfrac{x^2}{1 - x^2}$
10. $\dfrac{3x^2 + 4}{x^2 - 9}$
11. $\dfrac{x(x - 2)}{x - 1}$
12. $\dfrac{x^3 + 1}{x^2 - 1}$

[Hint: Let $u = x - 1$]

13. $x^4 - x^3$
14. $x^4 + x^3$
15. $x^4 - x^2$
16. $x^4 - 3x^2 + 2\sqrt{2}$
17. $-x^3 + x + 1$
18. $-x^3 + \frac{3}{2}x^2 + x + 1$
19. $x^3 + \frac{1}{4}x^2 - \frac{7}{8}x - \frac{3}{8}$
20. $x^3 + 3x^2 + 2x$
21. $\dfrac{x^3}{2 - x^2} + x$
22. $\dfrac{x^2 + 1}{(x - 1)(x + 1)^2}$
23. $\dfrac{x^2}{2 + x}$
24. $x^2 + \dfrac{1}{1 - x}$
25. $x^3 + 7x^2 - 2x + 10$
26. $3x^3 + x^2 + 1$
27. $\dfrac{1 - x^2}{1 + x^2}$
28. $\dfrac{x}{1 - x}$
29. $8x^3 - 3x^2 + 2x$
30. $x^4 - 6x^2 + 5x + 2$

[*Hint*: Sketch the derivative first.]

31. Match the following functions with the graphs in Fig. 3.4.21.
    (a) $\dfrac{x^2 + 1}{x^2 - 1}$,
    (b) $x + \dfrac{1}{x}$,
    (c) $x^3 - 3x^2 - 9x + 1$,
    (d) $\dfrac{x^2 - 1}{x^2 + 1}$.

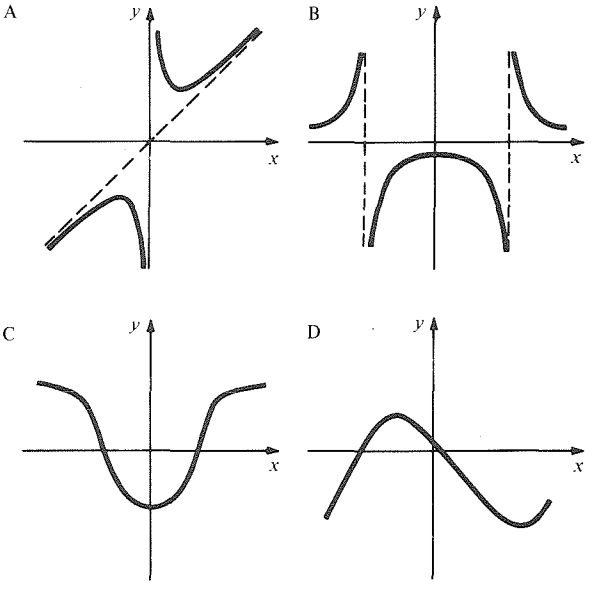


**Figure 3.4.21.** Match these graphs with the functions in Exercise 31.

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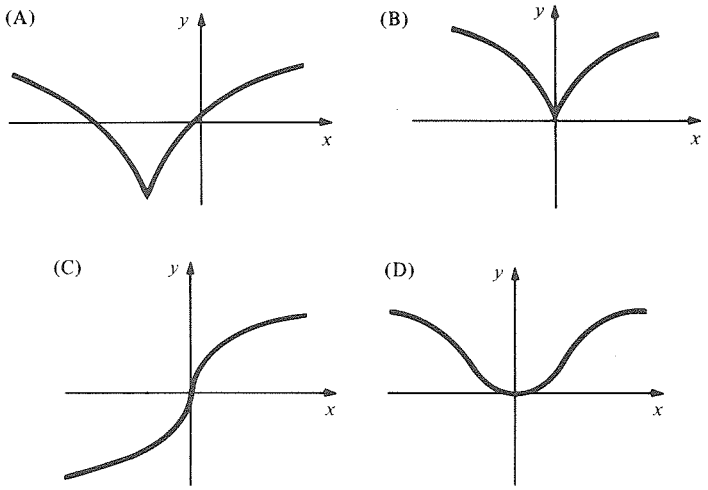


**Figure 3.4.22.** Match these graphs with the functions in Exercise 32.

32. Match the following functions with the graphs in Fig. 3.4.22: (a) $x^{2/7}$, (b) $x^{3/7}$, (c) $(1+x)^{2/7}-1$, (d) $(1+x^2)^{1/7}-1$.
33. Let $f(x)=(x^2-3)^{2/3}$. (a) Where is $f$ increasing? Decreasing? (b) Sketch the graph of $f$, noting any cusps.
34. Show that the graph of
$$f(x)=(1-x)^{2/3}(1+x)^{2/3}$$
has two cusps. Sketch.

Sketch the graph of each of the functions in Exercises 35–40.

35. $(x^2+1)^{1/3}$
36. $(x^2+5x+4)^{2/7}$
37. $(x-1)^{4/3}(x+1)^{2/3}$
38. $x+x^{2/3}$
39. $(x-4)^{100/99}$
40. $(x^3+2x^2+x)^{4/9}$.

41. Let $f(x)$ be a polynomial. Show that $f(x)$ is an even (odd) function if only even (odd) powers of $x$ occur with nonzero coefficients in $f(x)$.
42. Find a criterion for telling when the quotient $f(x)/g(x)$ of two polynomials is even, odd, or neither.
43. A simple model for the voting population in a certain district is given by $N(t)=30+12t^2-t^3$, $0\leqslant t\leqslant 8$, where $t$ is the time in years, $N$ the population in thousands.
    (a) Graph $N$ versus $t$ on $0\leqslant t\leqslant 8$.
    (b) At what time $t$ will the population of voters increase most rapidly?
    (c) Explain the significance of the points $t=0$ and $t=8$.
44. The population $P$ of mice in a wood varies with the number $x$ of owls in the wood according to the formula $P=30+10x^2-x^3$, $0\leqslant x\leqslant 10$. Graph $P$ versus $x$.

★45. Suppose that $f(x)$ is defined on all of $(-\infty,\infty)$. Show that
$$f(x)=e(x)+o(x),$$
where $e$ is an even function and $o$ is an odd function. [*Hint*: Substitute $-x$ for $x$, use the fact that $e$ is even and $o$ is odd, and solve for $e(x)$ and $o(x)$.]

★46. There is one function which is both even and odd. What is it?

★47. If $f$ is twice differentiable and $x_0$ is a critical point of $f$, must $x_0$ be either a local maximum, local minimum, or inflection point?

★48. What does the graph of $[ax/(bx+c)]+d$ look like if $a$, $b$, $c$, and $d$ are positive constants?

★49. Prove that the graph of any cubic $f(x)=ax^3+bx^2+cx+d$ $(a\neq 0)$ is symmetric about its inflection point in the sense that the function
$$g(x)=f\left(x-\frac{b}{3a}\right)-f\left(-\frac{b}{3a}\right)$$
is odd.

★50. A drug is injected into a person's bloodstream and the temperature increase $T$ recorded one hour later. If $x$ milligrams are injected, then
$$T(x)=\frac{x^2}{8}\left(1-\frac{x}{16}\right),\qquad 0\leqslant x\leqslant 16.$$
    (a) The rate of change $T$ with respect to dosage $x$ is called the *sensitivity* of the body to the dosage. Find it.
    (b) Use the techniques for graphing cubics to graph $T$ versus $x$.
    (c) Find the dosage at which the sensitivity is maximum.

★51. Convert $x^3+x^2+3x+1$ to the form $x^3+cx$ and determine whether the cubic is of type I, II, or III.

★52. Convert $x^3-3x^2+3x+1$ to the form $x^3+cx$ and determine whether the cubic is of type I, II, or III.

★53. Suppose that an object has position at time $t$ given by $x=(t-1)^{5/3}$. Discuss its velocity and acceleration near $t=1$ with the help of a graph.

★54. Consider $g(x)=f(x)(x-a)^{p/q}$, where $f$ is differentiable at $x=a$, $p$ is even, $q$ is odd and $p<q$. If $f(a)\neq 0$, show that $g$ has a cusp at $x=a$. [*Hint*: Look at $g'(x)$ for $x$ on either side of $a$.]

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★55. For which values of $a$ and $p$ does the cubic $g(x) = ax^3 + px$ have zero, one, or two critical points? (Assume $a \neq 0$).

★56. This problem concerns the graph of the general cubic (see the Supplement to this Section):

(a) Find an explicit formula for the coefficient $c_2$ in $f(x - b_1/3)$ in terms of $b_1$, $c_1$, and $d_1$ and thereby give a simple rule for determining whether the cubic

$$x^3 + b_1x^2 + c_2x + d_1$$

is of type I, II, or III.

(b) Give a rule, in terms of $a$, $b$, $c$, $d$, for determining the type of the general cubic $ax^3 + bx^2 + cx + d$.

(c) Use the quadratic formula on the derivative of $ax^3 + bx^2 + cx + d$ to determine, in terms of $a$, $b$, $c$, and $d$, how many turning points there are. Compare with the result in part (b).

Exercises 57 to 64 concern the graph of the general quartic:

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \qquad a \neq 0.$$

★57. Using the substitution $x - b/4a$ for $x$, show that one can reduce to the case

$$f(x) = x^4 + cx^2 + dx + e$$

(with a new $c$, $d$ and $e$!).

★58. According to the classification of cubics, $f'(x) = 4x^3 + 2cx + d$ can be classified into three types: I ($c > 0$), II ($c = 0$), and III ($c < 0$), so we may name each quartic by the type of its derivative. Sketch the graph of a typical quartic of type I.

★59. Divide type II quartics into three cases: $\text{II}_1$ ($d > 0$), $\text{II}_2$ ($d = 0$), $\text{II}_3$ ($d < 0$). Sketch their graphs with $e = 0$.

★60. In case III ($c < 0$), $f'(x)$ has two turning points and can have one, two, or three roots. By considering Fig. 3.4.23, show that the sign of $f'(x)$ at its critical points determines the number of roots. Obtain thereby a classification of type III quartics into five subtypes $\text{III}_1$, $\text{III}_2$, $\text{III}_3$, $\text{III}_4$, and $\text{III}_5$. Sketch the graphs for each case and determine the conditions on $c$ and $d$ which govern the cases.

★61. Using your results from Exercises 59 and 60, show that the $(c, d)$ plane may be divided into regions, as shown in Fig. 3.4.24, which determine the type of quartic.[4]

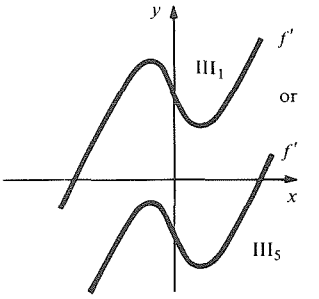
Using the results of Exercise 61, classify and sketch the graphs of the quartics in Exercises 62–64.

★62. $x^4 - 3x^2 - 4x$.

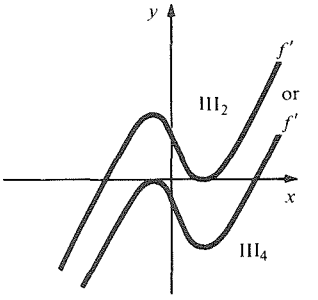
★63. $x^4 + 4x^2 + 6x$.

★64. $x^4 + 7x$.

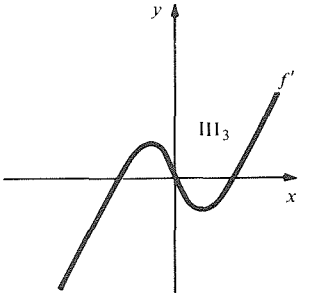
[4] For advanced (and sometimes controversial) applications of this figure, see T. Poston and I. Stuart, *Catastrophe Theory and Its Applications*, Pitman (1978).




(a) One root: ($f'$ has the same sign at its two critical points)




(b) Two roots: ($f'$ is zero at one of its critical points)




(c) Three roots: ($f'$ has opposite signs at its critical points)

**Figure 3.4.23.** The five possible positions of the graph of $f'$ in case III.

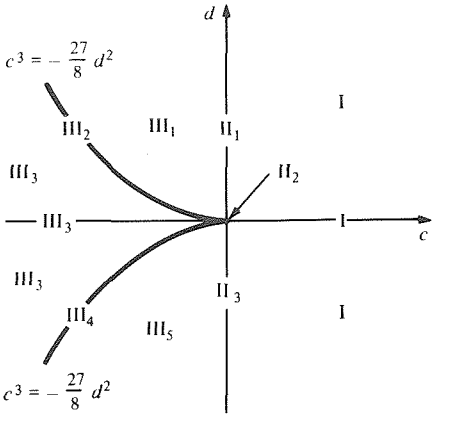



**Figure 3.4.24.** Locating the value of $(c, d)$ in this graph tells the type of quartic.

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# 3.5 Maximum–Minimum Problems

*A step-by-step procedure aids in the solution of practical maximum–minimum problems.*

This section is concerned with maximum–minimum problems of two types. First, we consider the problem of maximizing or minimizing a function on an interval. Then we apply these ideas to maximum–minimum problems that are presented in words rather than in formulas. Students are often overwhelmed by such "word problems," which appear to admit no systematic means of solution. Fortunately, guidelines do exist for attacking these problems; they are best learned through practice.

The maximum and minimum points discussed in Section 3.2 were *local*, since we compared $f(x_0)$ with $f(x)$ for $x$ *near* $x_0$. For many applications of calculus, however, it is important to find the points where $f(x)$ has the largest or smallest possible value as $x$ ranges over an entire given interval. In this section, we show how calculus helps us to locate these *global* maximum and minimum points, and we discuss how to translate word problems into calculus problems involving maxima and minima.

Global maxima and minima should be as familiar to you as the daily weather report. The statement on the 6 P.M. news that "today's high temperature was 26°C" means that:

1. At no time today was the temperature higher than 26°C.
2. At some time today, the temperature was exactly 26°C.

If we let $f$ be the function which assigns to each $t$ in the interval $[0, 18]$ the temperature in degrees at time $t$ hours after midnight, then we may say that 26 is the (global) *maximum value of $f$ on* $[0, 18]$. It is useful to have a formal definition.

### Maximum and Minimum Values on Intervals

Let $f$ be a function which is defined on an interval $I$ of real numbers.

If $M$ is a real number such that:

1. $f(x) \leqslant M$ for all $x$ in $I$
2. $f(x_0) = M$ for at least one $x_0$ in $I$

then we call $M$ the *maximum value of $f$ on $I$*.

If $m$ is a real number such that:

1. $f(x) \geqslant m$ for all $x$ in $I$
2. $f(x_1) = m$ for at least one $x_1$ in $I$

then we call $m$ the *minimum value of $f$ on $I$*.

The numbers $x_0$ and $x_1$ in the definition of maximum and minimum values represent the points at which these values are attained. They are called maximum or minimum *points* for $f$ on $I$. For the temperature function discussed above, $x_0$ might be 15.5, indicating that the high temperature

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occurred at 3:30 P.M. Of course, it might be possible that the temperature rose to 26° at 2 P.M. dipped due to a sudden rain shower, rose again to 26° at 3:30 P.M. and finally decreased toward evening. In that case both 14 and 15.5 would be acceptable values for $x_0$. (See Fig. 3.5.1.) Note that there were *local minima at* $t = 11$ (11 A.M.), *and* $t = 15$ (3 P.M.).

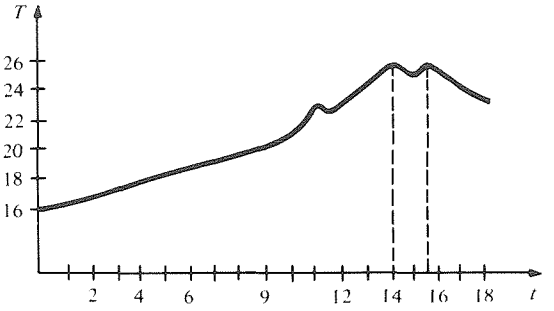


**Figure 3.5.1.** The maximum temperature (26°) occurred at 2 P.M. ($t = 14$) and also at 3:30 P.M. ($t = 15.5$).

Sometimes, to distinguish the points considered here from the local maxima and minima discussed in Section 3.2, we call them *global* maximum and minimum points.

We will see shortly that differential calculus provides a powerful technique for locating the maximum and minimum points of functions defined by formulas. For functions defined by other means, there are sometimes direct approaches to finding maxima and minima. For example:

1. The highest ring around a bathtub indicates the maximum water level achieved since the tub was last scrubbed. (Other rings indicate local maxima.)
2. On a maximum–minimum thermometer, one can read directly the maximum and minimum temperatures reached since the last time the thermometer was reset.
3. If the graph of a function is available, either from experimental data or by plotting from a formula as in Sections R.6 and 3.4, the maxima and minima can be seen as the high and low points on the graph.

**Example 1** Find the maximum and minimum points and values of $f(x) = 1/(1 + x^2)$ on the interval $[-2, 2]$.

**Solution** $f(x)$ is largest where its denominator is smallest, and vice versa. The maximum point occurs, therefore, at $x = 0$; the maximum value is 1. The minimum points occur at $-2$ and 2; the minimum value is $\frac{1}{5}$.

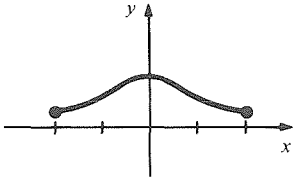


**Figure 3.5.2.** The function $f(x) = 1/(1 + x^2)$ on $[-2, 2]$ has a maximum point at $x = 0$ and minimum points at $x = \pm 2$.

We may verify these statements with the assistance of some calculus: $f'(x) = -2x/(1 + x^2)^2$, which is positive for $x < 0$ and negative for $x > 0$, so $f$ is increasing on $[-2, 0]$ and decreasing on $[0, 2]$. It follows that

$$\tfrac{1}{5} = f(-2) < f(x) < f(0) = 1 \qquad \text{for} \quad -2 < x < 0$$

and

$$1 = f(0) > f(x) > f(2) = \tfrac{1}{5} \qquad \text{for} \quad 0 < x < 2,$$

and we have $\frac{1}{5} \leqslant f(x) \leqslant 1$ for $-2 \leqslant x \leqslant 2$. (See Fig. 3.5.2.) ▲

**Example 2** Find the maximum and minimum points and values, if they exist, for the function $f(x) = x^2 + 1$ on each of the following intervals:

(a) $(-\infty, \infty)$ (c) $(-1, 1)$ (e) $(0, 1]$ (g) $(-2, 1]$
(b) $(0, \infty)$ (d) $[-1, 1]$ (f) $[\frac{1}{2}, 1]$ (h) $[-2, 1)$

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**Solution** The graphs are indicated in Fig. 3.5.3. In case (a), there is no highest point on the graph; hence, no maximum. In case (c), one might be tempted to call 2 the maximum value, but it is not attained at any point of the interval $(-1, 1)$. Study this example well; it will be a useful test case for the general statements to be made later in this section. ▲

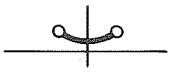
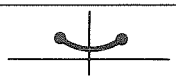
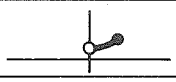
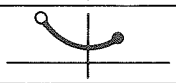
| | *I* | *Graph* | *Maximum points* | *Maximum value* | *Minimum points* | *Minimum value* |
|---|---|---|---|---|---|---|
| (a) | $(-\infty, \infty)$ | | None | None | 0 | 1 |
| (b) | $(0, \infty)$ | | None | None | None | None |
| (c) | $(-1, 1)$ |  | None | None | 0 | 1 |
| (d) | $[-1, 1]$ |  | $-1, 1$ | 2 | 0 | 1 |
| (e) | $(0, 1]$ |  | 1 | 2 | None | None |
| (f) | $[\frac{1}{2}, 1]$ | | 1 | 2 | $\frac{1}{2}$ | $\frac{5}{4}$ |
| (g) | $(-2, 1]$ |  | None | None | 0 | 1 |
| (h) | $[-2, 1)$ | | $-2$ | 5 | 0 | 1 |

**Figure 3.5.3.** Solutions to Example 2.

→ means that the graph goes off to infinity

O means that the endpoint does not belong to the graph

**Example 3** Figure 3.5.4 shows the amount of solar energy received at various latitudes in the northern hemisphere on June 21 on a square meter of horizontal surface located at the top of the atmosphere. Find the maximum and minimum points and values.

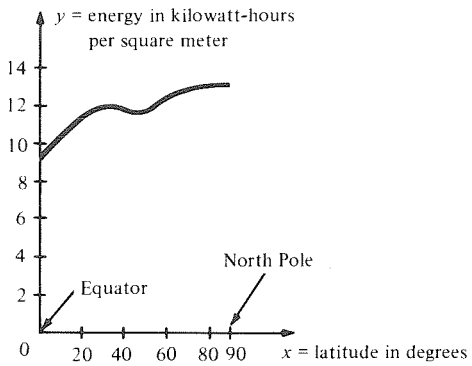



**Figure 3.5.4.** The solar energy $y$ received on June 21 at the top of the atmosphere at various latitudes $x$. (See W. G. Kendrew, *Climatology*, Oxford University Press, 1949.)

**Solution** The maximum value is about 13 kilowatt-hours, attained at $x = 90$ (the North Pole); the minimum value is about 9 kilowatt-hours, attained at $x = 0$ (the equator). (There are local maximum and minimum points at about $x = 30$ and $x = 50$, but these are not what we are looking for.) ▲

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The explanation for the unexpected result in Example 3 is that although the sun's radiation is weakest at the pole, the summer day is longest there, resulting in a larger amount of accumulated energy. At the surface of the earth, absorption of solar energy by the atmosphere is greatest near the pole, since the low angle of the sun makes the rays pass through more air. If this absorption is taken into account, the resulting graph is more like the one in Fig. 3.5.5, which is in better correspondence with the earth's climate.

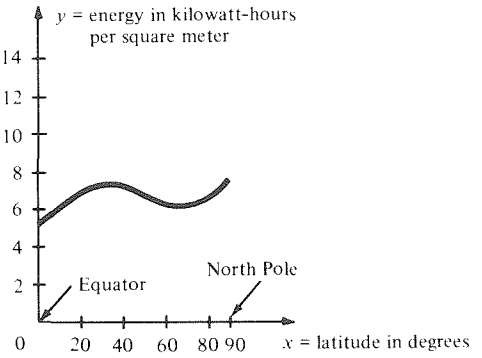


**Figure 3.5.5.** The solar energy received on June 21 at the surface of the earth, assuming clear skies. (See W. G. Kendrew, *Climatology*, Oxford University Press, 1949.)

If $I$ is a closed interval, the problem of finding maxima and minima for a function $f$ on $I$ is guaranteed to have a solution by the following theoretical result.

**Extreme Value Theorem**

*Let $f$ be continuous on the closed interval $[a,b]$. Then $f$ has both a maximum and a minimum value on $[a,b]$.*

The proof of this theorem is omitted,[4] but the statement should be understood by everyone. It says that two conditions together are *sufficient* to ensure that $f$ has both a maximum and a minimum value on $I$:

1. $f$ is continuous on $I$.
2. $I$ is a closed interval.

Cases (a), (b), (c), (e), and (g) of Example 2 show that condition 1 alone is not sufficient; that is, if $I$ is not closed, the maxima and minima may not exist. Case (h) shows that the maxima and minima might happen to exist even if $I$ is not closed; thus these two conditions are not *necessary* for the existence of maxima and minima.

Notice that, like the intermediate value theorem, the extreme value theorem is an *existence theorem* which tells you nothing about how to find the maxima and minima. However, combining the extreme value theorem with the critical point test does yield a practical test.

To obtain this test, we first note that by the extreme value theorem, the maximum and minimum points must exist; by the critical point test (Section 3.2), these points must be either critical points or endpoints. It remains, therefore, to determine *which* amongst the critical points and endpoints are the maximum and minimum points; to do this, it suffices to evaluate $f$ at all of them and then compare the values. The following box summarizes the situation.

[4] The proof may be found in any of the references listed in the Preface.

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## Closed Interval Test

To find the maxima and minima for a function $f$ which is continuous on $[a,b]$ and differentiable on $(a,b)$:

1. Make a list $x_1, \ldots, x_n$ consisting of the critical points of $f$ on $(a,b)$ and the endpoints $a$ and $b$ of $[a,b]$.
2. Compute the values $f(x_1), \ldots, f(x_n)$.

The largest of the $f(x_i)$ is the maximum value of $f$ on $[a,b]$. The maximum points for $f$ on $[a,b]$ are those $x_i$ for which $f(x_i)$ equals the maximum value.

The smallest of the $f(x_i)$ is the minimum value of $f$ on $[a,b]$. The minimum points for $f$ on $[a,b]$ are those $x_i$ for which $f(x_i)$ equals the minimum value.

**Example 4** Find the maximum and minimum points and values for the function $f(x) = (x^2 - 8x + 12)^4$ on the interval $[-10, 10]$.

**Solution** The list indicated by the closed interval test consists of $-10, 10$, and the critical points. To find the critical points we differentiate:

$$\frac{d}{dx}(x^2 - 8x + 12)^4 = 4(x^2 - 8x + 12)^3(2x - 8)$$

$$= 8(x-6)^3(x-2)^3(x-4)$$

which is zero when $x = 2$, 4, or 6. We compute the value of $f$ at each of these points and put the results in a table:

| $x$ | $-10$ | 2 | 4 | 6 | 10 |
|---|---|---|---|---|---|
| $f(x)$ | $(192)^4$ | 0 | $(-4)^4$ | 0 | $(32)^4$ |

The maximum value is $(192)^4 = 1358954496$; the maximum point is the endpoint $-10$. The minimum value is zero; the minimum points are the critical points 2 and 6. ▲

For intervals which are not closed, we may use graphing techniques to decide which critical points are maxima and minima.

**Example 5** Find maximum and minimum points and values for $y = x^4 - x^2$ on $[-1, \infty)$.

**Solution** The critical points satisfy the equation $0 = 4x^3 - 2x = 2x(2x^2 - 1)$. The solutions of the equation are 0, $\sqrt{1/2}$, and $-\sqrt{1/2}$, all of which lie in the interval $[-1, \infty)$. We make a table of values of the function

| $x$ | $-1$ | $-\sqrt{1/2}$ | 0 | $\sqrt{1/2}$ |
|---|---|---|---|---|
| $x^4 - x^2$ | 0 | $-\frac{1}{4}$ | 0 | $-\frac{1}{4}$ |

and its derivative

| $x$ | $-1$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 |
|---|---|---|---|---|---|
| $4x^3 - 2x$ | $-2$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 2 |

and draw a rough graph (Fig. 3.5.6).

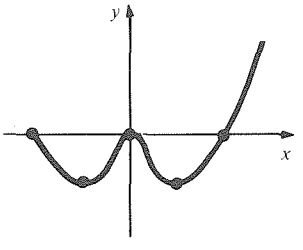


**Figure 3.5.6.** The graph of $y = x^4 - x^2$ on $[-1, \infty)$.

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We see that the critical points $-\sqrt{1/2}$ and $\sqrt{1/2}$ are minimum points; the minimum value is $-\frac{1}{4}$. The endpoint $-1$ and the critical point 0 are not maximum or minimum points since $x^4 - x^2$ takes on values both greater and less than 0 on the interval $[-1, \infty)$. Thus, there are no maximum points (0 is only a *local* maximum). ▲

We now turn our attention to solving maximum–minimum problems given in words rather than by formulas. To illustrate a general approach to these problems, we will go through the solution of four sample problems step by step.[5]

1. A shepherd lives on a straight coastline and has 500 meters of fencing with which to enclose his sheep. Assuming that he uses the coastline as one side of a rectangular enclosure, what dimensions should the rectangle have in order that the sheep have the largest possible area in which to graze?[6]
2. Illumination from a point light source is proportional to the intensity of the source and inversely proportional to the square of the distance from the source to the point of observation. Given two point sources 10 meters apart, with one source four times as intense as the other, find the darkest point on the line segment joining the sources.
3. Given four numbers, $a, b, c, d$, find a number $x$ which best approximates them in the sense that the sum of the squares of the differences between $x$ and each of the four numbers is as small as possible.
4. Suppose that it costs $(x^2/100) + 10x$ cents to run your car for $x$ days. Once you sell your car, it will cost you 50 cents a day to take the bus. How long should you keep the car?

The first thing to do is read the problem carefully. Then ask yourself: "What is given? What is required?"

Sometimes it appears that not enough data are given. In Problem 2, for instance, one may think: "Illumination is proportional to the intensity, but I'm not given the constant of proportionality, so the problem isn't workable." It turns out that in this problem, the answer does not depend upon the proportionality constant. (If it did, you could at least express your answer in terms of this unknown constant.) On the other hand, some of the data given in the statement of a problem may be irrelevant. You should do your best at the beginning of solving a problem to decide which data are relevant and which are not.

Here, in full, is the first step in attacking a maximum–minimum word problem.

### Step 1: Setting up the Problem

(a) Read the problem carefully, give names to any unnamed relevant variables, and note any relations among the variables.
(b) Draw a figure, if one is appropriate.
(c) Identify the quantity to be maximized or minimized.
(d) Make sure that the relevant and irrelevant information are clearly distinguished.

[5] For a general discussion of how to attack a problem, we enthusiastically recommend *How to Solve It*, by G. Polya (Princeton University Press, Second Edition, 1957).

[6] This ancient Greek problem is a variant of a famous problem ingeniously solved by Dido, the daughter of the king of Tyre and founder of Carthage (see M. Kline, *Mathematics: A Cultural Approach*, Addison-Wesley, 1962, p. 114).

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**Example 6** Carry out step 1 for Problems 1, 2, 3, and 4.

**Solution** *Problem* 1: We draw a picture (Fig. 3.5.7). Let $l$ and $w$ denote the length and width of the rectangle, and let $A$ be the area enclosed. These quantities are related by the equations $lw = A$ and $l + 2w = 500$ (since there are 500 meters of fencing available). We want to maximize $A$.

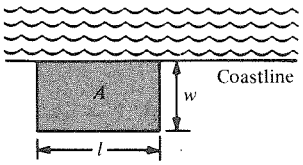


**Figure 3.5.7.** For which shape is $A$ largest?

*Problem* 2: We place the first source at $x = 0$ and the second source at $x = 10$ on the real line (Fig. 3.5.8). Denote by $I_1$ and $I_2$ the intensities of the two sources. Let $L_1$ = illumination at $x$ from the first source and $L_2$ = illumination at $x$ from the second source. Then $L = L_1 + L_2$ is the total illumination.

The given relations are

$$I_2 = 4I_1,$$

$$L_1 = \frac{kI_1}{x^2} \qquad (k > 0 \text{ is a proportionality constant}),$$

$$L_2 = \frac{kI_2}{(10 - x)^2}.$$

We want to minimize $L$.

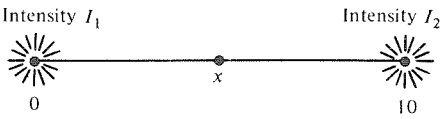


**Figure 3.5.8.** Light sources of intensities $I_1$ and $I_2$ are placed at $x = 0$ and $x = 10$. The observer is at $x$.

*Problem* 3: Call the unknown number $x$. We want to minimize

$$y = (x - a)^2 + (x - b)^2 + (x - c)^2 + (x - d)^2.$$

*Problem* 4: $x$ is the number of days we run the car. At this point in solving the problem, it is completely legitimate to pace around the room, muttering "What should be minimized?" This is not clearly stated in the problem, so we must determine it ourselves. A reasonable objective is to minimize the total amount of money to be paid. How is this to be done? Well, as soon as the cost of running the car exceeds 50 cents per day, we should switch to the bus. So let $y$ = the cost per day of running the car at day $x$. We want the first $x$ for which $y \geqslant 50$. The relation between the variables is

$$y = \frac{d}{dx}\left(\frac{x^2}{100} + 10x\right) = \frac{x}{50} + 10. \blacktriangle$$

Having set up a problem, we are ready to apply the methods of calculus.

## Step 2: Solving the Problem

(a) Write the quantity to be maximized or minimized as a function of one of the other variables in the problem. (This is usually done by expressing all other variables in terms of the one chosen.)
(b) Note any restrictions on the chosen variable.
(c) Find the maxima and minima by the methods of this chapter.
(d) State the answer in words.

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The main thing to be mastered in word problems is the technique of translating words into relevant mathematical symbols to which the tools of calculus can be applied. Once the calculus work is done, the answer must then be translated back into the terms of the original word problem.

**Example 7** Carry out step 2 for Problems 1, 2, 3, and 4.

**Solution** *Problem* 1: We want to maximize $A$, so we write it as a function of $l$ or $w$; we choose $w$. Now $l + 2w = 500$, so $l = 500 - 2w$. Thus $A = lw = (500 - 2w)w = 500w - 2w^2$.

The restriction on $w$ is that $0 \leqslant w \leqslant 250$. (Clearly, only a non-negative $w$ can be meaningful, and $w$ cannot be more than 250 or else $l$ would be negative.)

To maximize $A = 500w - 2w^2$ on $[0, 250]$, we compute $dA/dw = 500 - 4w$, which is zero if $w = 125$. Since $d^2A/dw^2 = -4$ for all $w$, the second derivative test tells us that 125 is a maximum point. Hence the maximum occurs when $w = 125$ and $l = 250$.

The rectangle should be 250 meters long in the direction parallel to the coastline and 125 meters in the direction perpendicular to the coastline in order to enclose the maximal area.

*Problem* 2: We want to minimize

$$L = L_1 + L_2 = kI_1\left[\frac{1}{x^2} + \frac{4}{(10-x)^2}\right].$$

Since the point is to be between the sources, we must have $0 < x < 10$. To minimize $L$ on $(0, 10)$, we compute:

$$L'(x) = kI_1\left[-\frac{2}{x^3} + \frac{4(20-2x)}{(10-x)^4}\right]$$

$$= 2kI_1\left[-\frac{1}{x^3} + \frac{4}{(10-x)^3}\right].$$

The critical points occur when

$$\frac{4}{(10-x)^3} = \frac{1}{x^3}.$$

Hence

$$\frac{10-x}{x} = \sqrt[3]{4},$$

$$10 - x - \sqrt[3]{4}\cdot x = 0,$$

$$x = \frac{10}{1+\sqrt[3]{4}}.$$

Thus there is one critical point; we use the first derivative test to determine if it is a maximum or minimum point. It suffices to check the sign of the derivative at a point on each side of the critical point.

Let $x_- = 0.0001$. Then

$$L'(x_-) = 2kI_1\left[-\frac{1}{(0.0001)^3} + \frac{4}{(10-0.0001)^3}\right].$$

Without calculating this explicitly, we can see that the term $-(1/0.0001)^3$ is negative and much larger in size than the other term, so $L'(x_-) < 0$.

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Let $x_+ = 9.9999$. Then

$$L'(x_+) = 2kI_1\left[-\frac{1}{(9.9999)^3} + \frac{4}{(0.0001)^3}\right].$$

Now it is the second term which dominates; since it is positive, $L'(x_+) > 0$. Thus $L'$ changes from negative to positive at $x_0$, so $x_0 = 10/(1 + \sqrt[3]{4})$ must be a minimum point.

The darkest point is thus at a distance of $10/(1 + \sqrt[3]{4}) \approx 3.86$ meters from the smaller source. It is interesting to compare the distances of the darkest point from the two sources. The ratio

$$\frac{10 - \left[10/(1 + \sqrt[3]{4})\right]}{10/(1 + \sqrt[3]{4})}$$

is simply $\sqrt[3]{4}$.

*Problem* 3: $y = (x - a)^2 + (x - b)^2 + (x - c)^2 + (x - d)^2$. There are no restrictions on $x$.

$$\frac{dy}{dx} = 2(x - a) + 2(x - b) + 2(x - c) + 2(x - d)$$
$$= 2(4x - a - b - c - d)$$

which is 0 only if $x = \frac{1}{4}(a + b + c + d)$.

Since $d^2y/dx^2 = 8$ is positive, $\frac{1}{4}(a + b + c + d)$ is the minimum point. The number required is thus the average, or arithmetic mean, of $a$, $b$, $c$, and $d$.

*Problem* 4: This is *not* a standard maximum–minimum problem. We minimize our expenses by selling the car at the time when the cost per day, $(x/50) + 10$, reaches the value 50. (See Fig. 3.5.9.) We solve $(x/50) + 10 = 50$, getting $x = 2000$. Thus the car should be kept for 2000 days. ▲

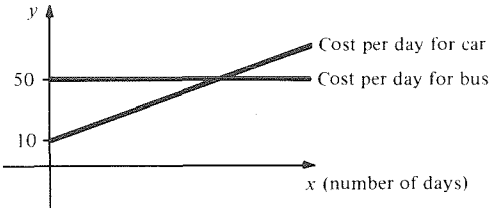


**Figure 3.5.9.** When does the car become more expensive than the bus?

In the process of doing a word problem, it is useful to ask general questions like, "Can I guess any properties of the answer? Is the answer reasonable?" Sometimes a clever or educated guess can carry one surprisingly far toward the solution of a problem.

For instance, consider the problem "Find the triangle with perimeter 1 which has the greatest area." In the statement of the problem, all three sides of the triangle enter in the same way—there is nothing to single out any side as special. Therefore, we guess that the answer must have the three sides equal; that is, the triangle should be equilateral. This is, in fact, correct. Such reasoning must be used with care: if we ask for the triangle with perimeter 1 and the *least* area, the answer is *not* an equilateral triangle; thus, reasoning by symmetry must often be supplemented by a more detailed analysis.

In Problems 1 and 3, the answers have as much symmetry as the data,

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and indeed these answers might have been guessed before any calculation had been done. In Problem 2, most people would have a hard time guessing the answer, but at least one can observe that the final answer has the darkest point nearer to the weaker of the two sources, which is reasonable.

**Example 8** Of all rectangles inscribed in a circle of radius 1, guess which has the largest area; the largest perimeter. (See Fig. 3.5.10.)

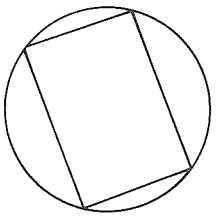

**Figure 3.5.10.** What shape gives the rectangle the largest area? The largest perimeter?

**Solution** Since the length and width enter symmetrically into the formulas for area and perimeter, we may guess that the maximum of both area and perimeter occurs when the rectangle is a square. ▲

We can now solve word problems by bringing together all of the preceding techniques.

**Example 9** Find the dimensions of a rectangular box of minimum cost if the manufacturing costs are 10 cents per square meter on the bottom, 5 cents per square meter on the sides, and 7 cents per square meter on the top. The volume is to be 2 cubic meters and the height is to be 1 meter.

**Solution** Let the dimensions of the base be $l$ and $w$; the height is 1. If the total cost is $C$, then

$$C = 10lw + 7lw + 2 \cdot 5 \cdot (l \cdot 1 + w \cdot 1) = 17lw + 10l + 10w.$$

Now $l \cdot w \cdot 1 = 2$ is the total volume. Eliminating $w$,

$$C = 34 + 10l + \frac{20}{l}.$$

We are to minimize $C$. Let $f(l) = 34 + 10l + (20/l)$ on $(0, \infty)$. Then

$$f'(l) = 10 - \frac{20}{l^2}$$

which is 0 when $l = \sqrt{2}$. (We are concerned only with $l > 0$.) Since $f''(l) = 40/l^3$ is positive at $l = \sqrt{2}$, $l = \sqrt{2}$ is a local minimum point. Since this is the only critical point, it is also the global minimum. Thus, the dimensions of minimum cost are $\sqrt{2}$ by $\sqrt{2}$ by 1. ▲

**Example 10** The stiffness $S$ of a wooden beam of rectangular cross-section is proportional to its breadth and the cube of its thickness. Find the stiffest rectangular beam that can be cut from a circular log of diameter $d$.

**Solution** Let the breadth be $b$ and thickness be $t$. Then $S$ is proportional to $bt^3$. From Fig. 3.5.11, we have $(\frac{1}{2}b)^2 + (\frac{1}{2}t)^2 = (d/2)^2$, i.e., $b = \sqrt{d^2 - t^2}$ $(0 < t < d)$. To maximize $f(t) = bt^3 = t^3\sqrt{d^2 - t^2}$, we note that $f'(t) = t^2(3d^2 - 4t^2)/\sqrt{d^2 - t^2} = 0$ if $3d^2 - 4t^2 = 0$, i.e., $t = \sqrt{3}\,d/2$. Since $f'(t)$ has the same sign as $3d^2 - 4t^2$, which changes from positive to negative at $t = \sqrt{3}\,d/2$, $t = \sqrt{3}\,d/2$ is a maximum point; so the dimensions of the beam should be $t = \sqrt{3}\,d/2$ and $b = d/2$. ▲

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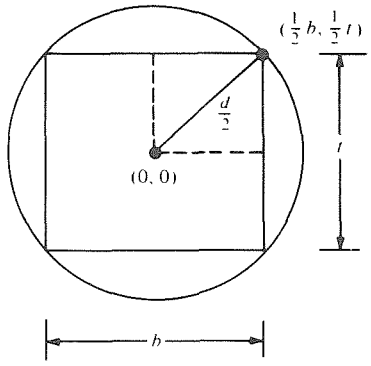


**Figure 3.5.11.** Cross section of a beam being cut from a log.

**Example 11** The terms marginal revenue and marginal cost were defined in Section 2.1.

(a) Prove that at a production level $x_0$ that maximizes profit, the marginal revenue equals the marginal cost (see Fig. 3.5.12).

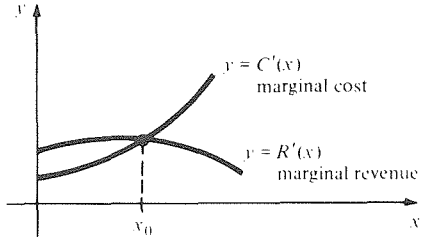


**Figure 3.5.12.** The production level $x_0$ that maximizes profit has $C'(x_0) = R'(x_0)$.

(b) Determine the number of units of a commodity that should be produced to maximize the profit when the cost and revenue functions are given by $C(x) = 800 + 30x - 0.01x^2$ and $R(x) = 50x - 0.02x^2$, $0 \leqslant x \leqslant 1500$.

**Solution** (a) We have $P(x) = R(x) - C(x)$. At as maximum point $x_0$ for $P$, we must have $P'(x_0) = 0$ by the critical point test. Therefore, $R'(x_0) - C'(x_0) = 0$ or $R'(x_0) = C'(x_0)$ as required.

(b) The profit is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (50x - 0.02x^2) - (800 + 30x - 0.01x^2) \\ &= 20x - 0.01x^2 - 800. \end{aligned}$$

The limits on $x$ are that $0 \leqslant x \leqslant 1500$. The critical points occur where $P'(x) = 0$; i.e., $20 - 0.02x = 0$, i.e., $x = 1000$. This is a maximum since $P''(x) = -0.02$ is negative; at $x = 1000$, $P(1000) = 9200$. The endpoint $x = 0$ is not a maximum since $P(0) = -800$ is negative. Also $P(1500) = 6700$, so the value at this endpoint is less than at $x = 1000$. Thus, the production level should be set at $x = 1000$. ▲

**Example 12** Given a number $a > 0$, find the minimum value of $(a + x)/\sqrt{ax}$ where $x > 0$.

**Solution** We are to minimize $f(x) = (a + x)/\sqrt{ax}$ on $(0, \infty)$. By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{\sqrt{ax} - (a + x)(a/2\sqrt{ax})}{ax} \\ &= \frac{2ax - a(a + x)}{2(ax)^{3/2}} = \frac{a(x - a)}{2(ax)^{3/2}}. \end{aligned}$$

Thus, $x = a$ is the only critical point in $(0, \infty)$. We observe that $f'(x)$ changes

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sign from negative to positive at $x = a$, so this is a minimum point. The minimum value is $f(a) = (a + a)/\sqrt{a^2} = 2$. ▲

The result of this example can be rephrased by saying that

$$\frac{a + x}{\sqrt{ax}} \geqslant 2 \qquad \text{for every} \quad a > 0, \quad x > 0.$$

Writing $b$ for $x$, this becomes

$$\frac{a + b}{2} \geqslant \sqrt{ab} \qquad \text{for every} \quad a > 0, \quad b > 0.$$

That is, the *arithmetic mean* $(a + b)/2$ of $a$ and $b$ is larger than their *geometric mean* $\sqrt{ab}$. This inequality was proved by using calculus. It can also be proved by algebra alone. Indeed, since the square of any number is non-negative,

$$0 \leqslant \left(\sqrt{a} - \sqrt{b}\right)^2 = a - 2\sqrt{ab} + b,$$

so $2\sqrt{ab} \leqslant a + b$. Hence,

$$\frac{a + b}{2} \geqslant \sqrt{ab}.$$

This is sometimes called the *arithmetic–geometric mean inequality*.

## Exercises for Section 3.5

Find the maximum and minimum points and values for each function on the given interval in Exercises 1–4.

1. $2x^3 - 5x + 2$ on $[1, \infty)$
2. $x^3 - 6x + 3$ on $(-\infty, \infty)$
3. $\dfrac{x^2 + 2}{x}$ on $(0, 5]$
4. $-3x^2 + 2x + 1$ on $(-\infty, \infty)$

Find the maximum and minimum points for each of the functions in Exercises 5–8.

5. $\dfrac{x^2 + 1}{x}$ on $(-\infty, 0)$
6. $\dfrac{x^2 + 2x + 3}{x^2 + 5}$ on $(-2, 2]$
7. $-x^3 + 5x + 4$ on $(-2, \frac{5}{3})$
8. $x^3 - 3x + 5$ on $(-3, \frac{3}{2})$
9. Figure 3.5.13 shows the annual inflation rate in Oxbridge for 1970–1980. Find the approximate maximum and minimum points and values.

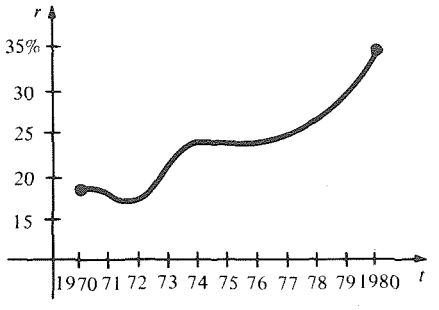


**Figure 3.5.13.** The annual inflation rate in Oxbridge.

10. Figure 3.5.14 shows the temperatures recorded in Goose Brow during a 24-hour period. When did the maxima and minima occur, and what were their values?

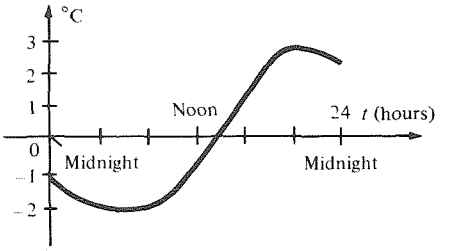


**Figure 3.5.14.** A cold day in Goose Brow.

In Exercises 11–30, find the critical points, endpoints, and global maximum and minimum points and values for each function on the given interval.

11. $f(x) = x^2 - x$ on $[0, 1)$.
12. $f(x) = x^3$ on $[-1, \infty)$.
13. $f(x) = x^4 - 4x^2 + 7$ on $[-4, 2]$.
14. $f(x) = 4x^4 - 2x^2 + 1$ (a) on $[-10, 20]$ and (b) on $[-0.2, -0.1]$.
15. $f(x) = \dfrac{1}{4x^4 - 2x^2 + 200{,}000}$ on $[-10, 20]$.
16. $f(x) = \dfrac{1 + x^2}{1 - x^2}$ on $[-\frac{1}{2}, \frac{3}{4}]$.
17. $f(x) = x^2 - 3x + 1$ on each of the following intervals.
    (a) $(2, \infty)$ (c) $(-3, 2]$
    (b) $(-\infty, \frac{1}{2}]$ (d) $(-\frac{3}{2}, 2]$

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(e) $(-2, 2)$ (g) $[-1, 1]$
(f) $(-\infty, \infty)$ (h) $[-8, 8]$

18. $f(x) = x^3 - 2x + 1$ on each of the intervals in Exercise 17.
19. $f(x) = 7x^2 + 2x + 4$ on (a) $[-1, 1]$, (b) $(0, \infty)$, (c) $[-4, 2)$.
20. $f(x) = x^3 + 6x^2 - 12x + 7$ on (a) $(-\infty, \infty)$, (b) $(-2, 6]$, (c) $[-2, 1)$, (d) $[-4, 4]$.
21. $f(x) = x^3 - 3x^2 + 3x + 1$ on $(-\infty, \infty)$.
22. $f(x) = x^3 + 3x^2 - 3x + 1$ on $[-1, 2]$.
23. $f(x) = x^3 - x$ on $[-2, 3]$.
24. $f(x) = x^4 + 8x^3 + 3$ on $[-1, 1]$.
25. $f(x) = \dfrac{x^3 - 1}{x^2 + 1}$ on $[-10, 10]$.
26. $f(x) = \dfrac{x^2 + 1}{x^4 + 1}$ on $(-\infty, \infty)$.
27. $f(x) = -\dfrac{x}{1 + x^2}$ on $[-1, 6]$.
28. $f(x) = \dfrac{x}{1 + x^2}$ on $(-\infty, \infty)$.
29. $f(x) = \dfrac{x^3}{1 + x^2}$ on $(-\infty, \infty)$.
30. $f(x) = \dfrac{x^n}{1 + x^2}$ on $(-\infty, \infty)$, $n$ a positive integer.

Carry out step 1 for Exercises 31–34, (see Example 6).

31. Of all rectangles with area 1, which has the smallest perimeter?
32. Find the point on the arc of the parabola $y = x^2$ for $0 \leqslant x \leqslant 1$ which is nearest to the point $(0, 1)$. [*Hint*: Consider the *square* of the distance between points.]
33. Two point masses which are a fixed distance apart attract one another with a force which is proportional to the product of the masses. Assuming that the sum of the two masses is $M$, what must the individual masses be so that the force of attraction is as large as possible?
34. Ten miles from home you remember that you left the water running, which is costing you 10 cents an hour. Driving home at speed $s$ miles per hour costs you $6 + (s/10)$ cents per mile. At what speed should you drive to minimize the total cost of gas and water?
35. Carry out step 2 for Exercise 31.
36. Carry out step 2 for Exercise 32.
37. Carry out step 2 for Exercise 33.
38. Carry out step 2 for Exercise 34.
39. A rectangular box, open at the top, is to be constructed from a rectangular sheet of cardboard 50 centimeters by 80 centimeters by cutting out equal squares in the corners and folding up the sides. What size squares should be cut out for the container to have maximum volume?
40. A window in the shape of a rectangle with a semicircle on the top is to be made with a perimeter of 4 meters. What is the largest possible area for such a window?
41. (a) A can is to be made to hold 1 liter (= 1000 cubic centimeters) of oil. If the can is in the shape of a circular cylinder, what should the radius and height be in order that the surface area of the can (top and bottom and curved part) be as small as possible?
    (b) What is the answer in part (a) if the total capacity is to be $V$ cubic centimeters?
    (c) Suppose that the surface area of a can is fixed at $A$ square centimeters. What should the dimensions be so that the capacity is maximized?
42. Determine the number of units of a commodity that should be produced to maximize the profit when the cost and revenue functions are given by $C(x) = 700 + 40x - 0.01x^2$, $R(x) = 80x - 0.03x^2$.
43. Determine the number of units of a commodity that should be produced to maximize the profit for the following cost and revenue functions: $C(x) = 360 + 80x + .002x^2 + .00001x^3$, $R(x) = 100x - .0001x^2$.
44. If the cost of producing $x$ calculators is $C(x) = 100 + 10x + 0.01x^2$ and the price per calculator at production level $x$ is $P(x) = 26 - 0.1x$ (this is called the *demand* equation), what production level should be set in order to maximize profit?
45. The U.S. Post Office will accept rectangular boxes only if the sum of the length and girth (twice the width plus twice the height) is at most 72 inches. What are the dimensions of the box of maximum volume the Post Office will accept? (You may assume that the width and height are equal.)
46. Given $n$ numbers, $a_1, \ldots, a_n$, find a number $x$ which best approximates them in the sense that the sum of the squares of the differences between $x$ and the $n$ numbers is as small as possible.
47. One positive number plus the square of another equals 48. Choose the numbers so that their product is as large as possible.
48. Find the point or points on the arc of the parabola $y = x^2$ for $0 \leqslant x \leqslant 1$ which are nearest to the point $(0, q)$. Express your answer in terms of $q$. [*Hint*: Minimize the *square* of the distance between points.]
49. One thousand feet of fencing is to be used to surround two areas, one square and one circular. What should the size of each area be in order that the total area be (a) as large as possible and (b) as small as possible?
50. A forest can support up to 10,000 rabbits. If there are $x$ rabbits in the forest, each female can be expected to bear $\frac{1}{1000}(10{,}000 - x)$ bunnies in a year. What total population will give rise to the

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greatest number of newborn bunnies in a year? (Assume that exactly half the rabbits are female; ignore the fact that the bunnies may themselves give birth to more young during the year; and remember that the total population including new bunnies is not to exceed 10,000).

51. (a) In Fig. 3.5.15, for which value of $y$ does the line segment $PQ$ have the shortest length? Express your answer in terms of $a$ and $b$. [*Hint*: Minimize the square of the length.]

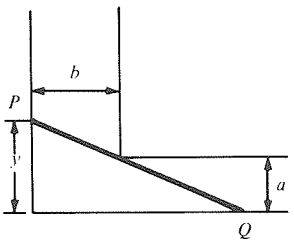


**Figure 3.5.15.** The "ladder" $PQ$ just fits into the corner of the corridor.

(b) What is the length of the longest ladder which can be slid along the floor around the corner from a corridor of width $a$ to a corridor of width $b$?

52. One hundred feet of fencing is to be used to enclose two pens, one square and one triangular. What dimensions should the pens have to enclose the largest possible area?
53. A conical dunce cap is to be made from a circular piece of paper of circumference $c$ by cutting out a pie-shaped piece whose curved outer edge has length $l$. What should $l$ be so that the resulting dunce cap has maximum volume?
54. (a) Suppose that you drive from coast to coast on Interstate Route 80 and your altitude above sea level is $f(x)$ when you are $x$ miles from San Francisco. Discuss the critical points, endpoints, global maximum and minimum points and values, and local maximum and minimum points for $f(x)$.

(b) Do as in part (a) for a hike to the top of Mt. Whitney, where $x$ is the distance walked from your starting point.

In Exercises 55–59, try to guess the answer, or some part of the answer, by using some symmetry of the data.

55. Of all rectangles of area 1, which has the smallest perimeter?
56. Of all geometric figures with perimeter 1, which has the greatest area?
57. In Problem 1 (see Example 6), suppose that we allow the fencing to assume any shape, not necessarily rectangular, but still with one side along the shore requiring no fencing. What shape gives the maximum area? A formal proof is not required.
58. What is the answer to Problem 2 (see Example 6) if the two intensities are equal? If one of the intensities is eight times the other?
59. Of all *right* triangles of area 1, guess which one has the shortest perimeter. Which one, if any, has the longest perimeter?
60. Use calculus to show that the answer in Example 8 is correct.

★61. Find a function on $[-1, 1]$ which is continuous but which is not differentiable at its maximum point.

★62. Find a function defined on $[0, 1]$ which does not have a maximum value on $[0, 1]$.

★63. Find a function defined on $[-2, 2]$ which has neither a maximum value nor a minimum value on $[-2, 2]$.

★64. Find a function defined on $[-3, -1]$ which is continuous at $-3$ and $-1$, has a maximum value on $[-3, -1]$, but has no minimum value on $[-3, -1]$.

★65. Prove that the maximum value of $f$ on $I$ is unique; that is, show that if $M_1$ and $M_2$ both satisfy conditions 1 and 2 of the definition at the beginning of this section, then $M_1 = M_2$. [*Hint*: Show that $M_1 \leqslant M_2$ and $M_2 \leqslant M_1$.]

★66. Let the set $I$ be contained in the set $J$, and suppose that $f$ is defined on $J$ (and therefore defined on $I$). Show that if $m_I$ and $m_J$ are the minimum values of $f$ on $I$ and $J$, respectively, then $m_J \leqslant m_I$.

In Exercises 67–70, find the maximum and minimum values of the given function on the given interval.

★67. $f(x) = px + (q/x)$, on $(0, \infty)$ where $p$ and $q$ are nonzero numbers.

★68.

$$f(x) = \begin{cases} 5x, & x \leqslant 0 \\ -3x, & x \geqslant 0 \end{cases} \quad \text{on } [-1, 1].$$

★69. $f(x) = x/(1 - x^2)$ for $-1 < x \leqslant 0$ and $f(x) = x^3 - x$ for $0 \leqslant x < 2$, on $(-1, 2)$.

★70.

$$f(x) = \left\{ \begin{array}{ll} -x^4 - 1 & \text{for } -1 < x \leqslant 0 \\ -1 & \text{for } 0 \leqslant x \leqslant 1 \\ x - 2 & \text{for } 1 \leqslant x < \infty \end{array} \right\}$$

on $(-1, \infty)$.

★71. Three equal light sources are spaced at $x = 0, 1, 2$, along a line. At what points between the sources is the total illumination least? See Problem 2, Example 6. You should get a cubic equation for $(x - 1)^2$. Solve the equation numerically by using the method of bisection.

★72. What happens in Exercise 71 if there are four light sources instead of three? Can you guess the correct answer by using a symmetry argument without doing any calculation?

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★73. (This minimization problem involves no calculus.) You ran out of milk the day before your weekly visit to the supermarket, and you must pick up a container at the corner grocery store. At the corner store, a quart costs $Q$ cents and a half gallon costs $G$ cents. At the supermarket, milk costs $q$ cents a quart and $g$ cents a half gallon.

(a) If $Q = 45$, $G = 80$, $q = 38$, and $g = 65$, what size container should you buy at the corner grocery to minimize your eventual milk expense? (Assume that a quart will get you through the day.)

(b) Under what conditions on $Q$, $G$, $q$, and $g$ should you buy a half gallon today?

★74. The cost of running a boat is $10v^3$ dollars per mile where $v$ is its speed in still water. What is the most economical speed to run the boat upstream against a current of 5 miles per hour?

★75. Let $f$ and $g$ be defined on $I$. Under what conditions is the maximum value of $f + g$ on $I$ equal to the sum of the maximum values of $f$ and $g$ on $I$?

★76. Let $I$ be the set consisting of the whole numbers from 1 to 1000, and let $f(x) = 45x - x^2$. Find the maximum and minimum points and values for $f$ on $I$.

★77. Suppose that $f$ is continuous on $[a, b]$ and is differentiable and concave upward on the interval $(a, b)$. Show that the maximum point of $f$ is an endpoint.

## 3.6 The Mean Value Theorem

*If the derivative of a function is everywhere zero, then the function is constant.*

The mean value theorem is a technical result whose applications are more important than the theorem itself. We begin this section with a statement of the theorem, proceed immediately to the applications, and conclude with a proof which uses the idea of global maxima and minima.

The mean value theorem is, like the intermediate value and extreme value theorems, an *existence* theorem. It asserts the existence of a point in an interval where a function has a particular behavior, but it does not tell you how to find the point.

### Mean Value Theorem

*If $f$ is continuous on $[a,b]$ and is differentiable on $(a,b)$, then there is a point $x_0$ in $(a,b)$ at which*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

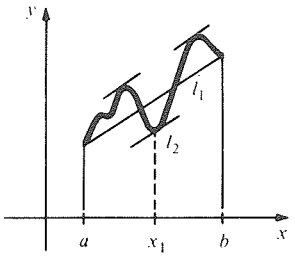


**Figure 3.6.1.** The slope $[f(b) - f(a)]/(b - a)$ of the secant line $l_1$ is equal to the slope of the tangent line $l_2$.

In physical terms, the mean value theorem says that the average velocity of a moving object during an interval of time is equal to the instantaneous velocity at some moment in the interval. Geometrically, the theorem says that a secant line drawn through two points on a smooth graph is parallel to the tangent line at some intermediate point on the curve. There may be more than one such point, as in Fig. 3.6.1. Consideration of these physical and geometric interpretations should make the theorem believable.

We will prove the mean value theorem at the end of this section. For now, we will concentrate on some applications. These tell us that if we know something about $f'(x)$ for all $x$ in $[a,b]$, then we can conclude something about the relation between values of $f(x)$ at different points in $[a,b]$.

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## Consequences of the Mean Value Theorem

Suppose that $f$ is differentiable on $(a,b)$ and continuous on $[a,b]$.

1. Assume that there are two numbers $A$ and $B$ such that
$$A \leqslant f'(x) \leqslant B \qquad \text{for all } x \text{ in } (a,b).$$
Then for any two distinct points $x_1$ and $x_2$ in $[a,b]$,
$$A \leqslant \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant B.$$
2. If $f'(x) = 0$ on $(a,b)$, then $f$ is constant on $[a,b]$.
3. Let $F(x)$ and $G(x)$ be functions such that $F'(x) = G'(x)$ for all $x$ in an open interval $(a,b)$. Then there is a constant $C$ such that $F(x) = G(x) + C$ for all $x$ in $(a,b)$.

The first consequence holds since $[f(x_2) - f(x_1)]/[x_2 - x_1] = f'(x_0)$ for some $x_0$ between $x_1$ and $x_2$, by the mean value theorem applied to $f$ on the interval with endpoints $x_1$ and $x_2$. Since $x_0$ is in $(a,b)$, $f'(x_0)$ lies between $A$ and $B$, so does the difference quotient. In particular, if $f'(x) = 0$, we can choose $A = B = 0$, which implies that $f(x_1) = f(x_2)$. Hence $f$ is constant. To obtain the third consequence, observe that $F(x) - G(x)$ has zero derivative, so it is a constant. This consequence of the mean value theorem is a fact about antiderivatives which we used in Section 2.5.

**Example 1** Suppose that $f$ is differentiable on the whole real line and that $f'(x)$ is constant. Prove that $f$ is linear.

**Solution** Let $m$ be the constant value of $f'$.

*Method* 1. We may apply the first consequence of the mean value theorem with $x_1 = 0$, $x_2 = x$, $A = m = B$, to conclude that $[f(x) - f(0)]/(x - 0) = m$. But then $f(x) = mx + f(0)$ for all $x$, so $f$ is linear.

*Method* 2. Let $g(x) = f(x) - mx$. Then $g'(x) = m - m = 0$, so $g$ is a constant. Setting $x = 0$, $g(x) = f(0)$. Thus $f(x) = mx + f(0)$ so $f$ is linear. ▲

**Example 2** Let $f$ be continuous on $[1,3]$ and differentiable on $(1,3)$. Suppose that for all $x$ in $(1,3)$, $1 \leqslant f'(x) \leqslant 2$. Prove that $2 \leqslant f(3) - f(1) \leqslant 4$.

**Solution** Apply the first consequence with $A = 1$, $B = 2$. Then we have the inequalities $1 \leqslant [f(3) - f(1)]/(3 - 1) \leqslant 2$, and so $2 \leqslant f(3) - f(1) \leqslant 4$. ▲

**Example 3** Let $f(x) = (d/dx)|x|$.

(a) Find $f'(x)$.

(b) What does the second consequence of the mean value theorem tell you about $f$? What does it not tell you?

**Solution** (a) Since $|x|$ is linear on $(-\infty, 0)$ and $(0, \infty)$, its second derivative $d^2|x|/dx^2 = f'(x)$ is identically zero for all $x \neq 0$.

(b) By consequence 2, $f$ is constant *on any open interval on which it is differentiable*. It follows that $f$ is constant on $(-\infty, 0)$ and $(0, \infty)$. The corollary does *not* say that $f$ is constant on $(-\infty, \infty)$. In fact, $f(-2) = -1$, while $f(2) = +1$. ▲

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**Example 4** Suppose that $F'(x) = x$ for all $x$ and that $F(3) = 2$. What is $F(x)$?

**Solution** Let $G(x) = \frac{1}{2}x^2$. Then $G'(x) = x = F'(x)$, so $F(x) = G(x) + C = \frac{1}{2}x^2 + C$. To evaluate $C$, set $x = 3$: $2 = F(3) = \frac{1}{2}(3^2) + C = \frac{9}{2} + C$. Thus $C = 2 - \frac{9}{2} = -\frac{5}{2}$ and $F(x) = \frac{1}{2}x^2 - \frac{5}{2}$. ▲

**Example 5** The velocity of a train is kept between 40 and 50 kilometers per hour during a trip of 200 kilometers. What can you say about the duration of the trip?

**Solution** Before presenting a formal solution using the mean value theorem, let us use common sense. If the velocity is at least 40 kilometers per hour, the trip takes at most $\frac{200}{40} = 5$ hours. If the velocity is at most 50 kilometers per hour, the trip takes at least $\frac{200}{50} = 4$ hours. Thus, the trip takes between 4 and 5 hours.

To use the mean value theorem, let $f(t)$ be the position of the train at time $t$; let $a$ and $b$ be the beginning and ending times of the trip. By consequence 1 with $A = 40$ and $B = 50$, we have $40 \leqslant [f(b) - f(a)]/(b - a) \leqslant 50$. But $f(b) - f(a) = 200$, so

$$40 \leqslant \frac{200}{b - a} \leqslant 50,$$

$$\frac{1}{5} \leqslant \frac{1}{b - a} \leqslant \frac{1}{4},$$

$$5 \geqslant b - a \geqslant 4.$$

Hence the trip takes somewhere between 4 and 5 hours, as we found above. ▲

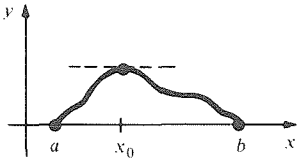


**Figure 3.6.2.** Rolle's theorem: If $f$ is zero at the ends of the interval, its graph must have a horizontal tangent line somewhere between.

Our proof of the mean value theorem will use two results from Sections 3.1 and 3.4, which we recall here:

1. If $x_0$ lies in the open interval $(a,b)$ and is a maximum or minimum point for a function $f$ on an interval $[a,b]$, and if $f$ is differentiable at $x_0$, then $f'(x_0) = 0$ (critical point test).
2. If $f$ is continuous on a closed interval $[a,b]$, then $f$ has a maximum and a minimum point in $[a,b]$ (extreme value theorem).

We now proceed with the proof in three steps.

**Step 1 (Rolle's Theorem[7])** *Let $f$ be continuous on $[a,b]$ and differentiable on $(a,b)$, and assume that $f(a) = f(b) = 0$. Then there is a point $x_0$ in $(a,b)$ at which $f'(x_0) = 0$.*

Indeed, if $f(x) = 0$ for all $x$ in $[a,b]$, we can choose any $x_0$ in $(a,b)$. So assume that $f$ is not everywhere zero. By the extreme value theorem, $f$ has a maximum point $x_1$ and a minimum point $x_2$. Since $f$ is zero at the ends of the interval but is not identically zero, at least one of $x_1, x_2$ lies in $(a,b)$, and not at an endpoint. Let $x_0$ be this point. By the critical point test, $f'(x_0) = 0$, so Rolle's theorem is proved.

Rolle's theorem has a simple geometric interpretation (see Fig. 3.6.2).

**Step 2 (Horserace Theorem)** *Suppose that $f_1$ and $f_2$ are continuous on $[a,b]$ and differentiable on $(a,b)$, and assume that $f_1(a) = f_2(a)$ and $f_1(b) = f_2(b)$. Then there exists a point $x_0$ in $(a,b)$ such that $f_1'(x_0) = f_2'(x_0)$.*

[7] Michel Rolle (1652–1719) (pronounced "roll") was actually best known for his attacks on the calculus. He was one of the critics of the newly founded theory of Newton and Leibniz. It is an irony of history that he has become so famous for "Rolle's theorem" when he did not even prove the theorem but used it only as a remark concerning the location of roots of polynomials. (See D. E. Smith, *Source Book in Mathematics*, Dover, 1929, pp. 251–260, for further information.)

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Let $f(x) = f_1(x) - f_2(x)$. Since $f_1$ and $f_2$ are differentiable on $(a, b)$ and continuous on $[a, b]$, so is $f$. By assumption, $f(a) = f(b) = 0$, so from Step 1, $f'(x_0) = 0$ for some $x_0$ in $(a, b)$. Thus $f_1'(x_0) = f_2'(x_0)$ as required (see Fig. 3.6.3).

We call this the "horserace theorem" because it has the following interpretation. Suppose that two horses run a race starting together and ending in a tie. Then, at some time during the race, they must have had the same velocity.

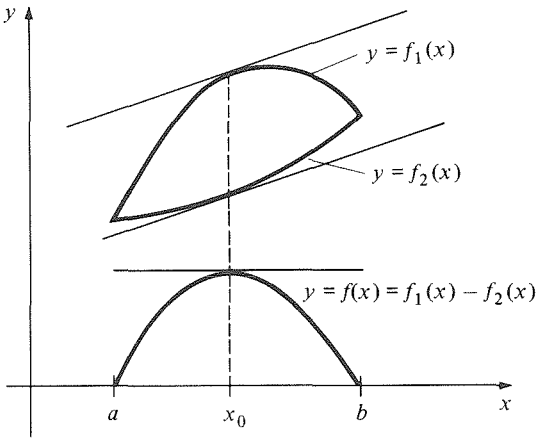


**Figure 3.6.3.** The curves $y = f_1(x)$ and $y = f_2(x)$ have parallel tangent lines when $y = f_1(x) - f_2(x)$ has a horizontal tangent line.

**Step 3.**

We apply Step 2 to a given function $f$ and a linear function $l$ that matches $f$ at its endpoints, namely,

$$l(x) = f(a) + (x - a)\left[\frac{f(b) - f(a)}{b - a}\right].$$

Note that $l(a) = f(a)$, $l(b) = f(b)$, and $l'(x) = [f(b) - f(a)]/(b - a)$. By Step 2, $f'(x_0) = l'(x_0) = [f(b) - f(a)]/(b - a)$ for some point $x_0$ in $(a, b)$.

This completes our proof of the mean value theorem. ■

**Example 6** Let $f(x) = x^4 - 9x^3 + 26x^2 - 24x$. Note that $f(0) = 0$ and $f(2) = 0$. Show without calculating that $4x^3 - 27x^2 + 52x - 24$ has a root somewhere strictly between 0 and 2.

**Solution** Since $f(0) = 0$ and $f(2) = 0$, Rolle's theorem shows that $f'$ is zero at some $x_0$ in $(0, 2)$; that is, $0 < x_0 < 2$. ▲

**Example 7** Suppose that $f$ is a differentiable function such that $f(0) = 0$ and $f(1) = 1$. Show that $f'(x_0) = 2x_0$ for some $x_0$ in $(0, 1)$.

**Solution** Use the horserace theorem with $f_1(x) = f(x)$, $f_2(x) = x^2$, and $[a, b] = [0, 1]$. ▲

## Exercises for Section 3.6

1. Suppose that $f$ is continuous on $[0, \frac{1}{2}]$, and differentiable on $(0, \frac{1}{2})$, and that $0.3 \leqslant f'(x) \leqslant 1$ for $0 < x < \frac{1}{2}$. Prove that $0.15 \leqslant [f(\frac{1}{2}) - f(0)] \leqslant 0.5$.
2. Suppose that $f$ is continuous on $[3, 5]$ and differentiable on $(3, 5)$, and that $\frac{1}{2} \leqslant f'(x) \leqslant \frac{2}{3}$ for $3 < x < 5$. Show that $1 \leqslant [f(5) - f(3)] \leqslant \frac{4}{3}$.
3. Suppose that $(d/dx)[f(x) - 2g(x)] = 0$. What can you say about the relationship between $f$ and $g$?
4. If $f''(x) = 0$ on $(a, b)$, what can you say about $f$?
5. Let $f(x) = x^5 + 8x^4 - 5x^2 + 15$. Prove that somewhere between $-1$ and 0 the tangent line to the graph of $f$ has slope $-2$.
6. Let $f(x) = 5x^4 + 9x^3 - 11x^2 + 10$. Prove that the graph of $f$ has slope 9 somewhere between $-1$ and 1.
7. Let $f(x) = \sqrt{x^3 - 8}$. Show that somewhere between 2 and 3 the tangent line to the graph of $f$ has slope $\sqrt{19}$.

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8. Let $f(x) = x^7 - x^5 - x^4 + 2x + 1$. Prove that the graph of $f$ has slope 2 somewhere between $-1$ and 1.
9. Suppose that an object lies at $x = 4$ when $t = 0$ and that the velocity $dx/dt$ is 35 with a possible error of $\pm 1$, for all $t$ in $[0, 2]$. What can you say about the object's position when $t = 2$?
10. The fuel consumption of an automobile varies between 17 and 23 miles per gallon, according to the conditions of driving. Let $f(x)$ be the number of gallons of fuel in the tank after $x$ miles have been driven. If $f(100) = 15$, give upper and lower estimates for $f(200)$.
11. Directly verify the validity of the mean value theorem for $f(x) = x^2 - x + 1$ on $[-1, 2]$ by finding the point(s) $x_0$. Sketch.
12. Let $f(x) = x^3$ on the interval $[-2, 3]$. Find explicitly the value(s) of $x_0$ whose existence is guaranteed by the mean value theorem. Sketch.
13. Suppose that $F'(x) = -(1/x^2)$ for all $x \neq 0$. Is $F(x) = 1/x + C$, where $C$ is a constant?
14. Suppose that $f'(x) = x^2$ and $f(1) = 0$. What is $f(x)$?
15. Let $f(x) = |x| - 1$. Then $f(-1) = f(1) = 0$, but $f'(x)$ is never equal to zero on $[-1, 1]$. Does this contradict Rolle's theorem? Explain.
16. Suppose that the horses in a race cross the finish line with equal velocities. Must they have had the same acceleration at some time during the race?

Find the antiderivatives of the functions in Exercises 17–20.

17. $f(x) = \frac{1}{2}x - 4x^2 + 21$.
18. $f(x) = 6x^5 - 12x^3 + 15x - 11$.
19. $f(x) = \dfrac{1}{x^2} + 2x$.
20. $f(x) = 2x(x^2 + 7)^{100}$.

Find the antiderivative $F(x)$ for the given function $f(x)$ satisfying the given condition in Exercises 21–24.

21. $f(x) = 2x^4$; $F(1) = 2$.
22. $f(x) = 4 - x$; $F(2) = 1$.
23. $f(x) = x^4 + x^3 + x^2$; $F(1) = 1$.
24. $f(x) = \dfrac{1}{x^5}$; $F(1) = 3$.
25. Let $f$ be twice differentiable on $(a, b)$, and suppose that $f(x) = 0$ at three distinct points in $(a, b)$. Prove that there is a point $x_0$ in $(a, b)$ at which $f''(x_0) = 0$.

★26. Use the mean value theorem to prove *the increasing function theorem*: If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, and $f'(x) > 0$ for $x$ in $(a, b)$, then $f$ is increasing on $[a, b]$.

★27. Suppose that $f$ and $g$ are continuous on $[a, b]$ and that $f'$ and $g'$ are continuous on $(a, b)$. Assume that

$$f(a) = g(a) \quad \text{and} \quad f(b) = g(b).$$

Prove that there is a number $c$ in $(a, b)$ such that the line tangent to the graph of $f$ at $(c, f(c))$ is parallel to the line tangent to the graph of $g$ at $(c, g(c))$.

★28. Let $f$ be a polynomial. Suppose that $f$ has a double zero at $a$ and $b$. (A polynomial $f$ has a *double zero* at $x = a$ if $f(x) = (x - a)^2 g(x)$ for some polynomial $g$). Show that $f'(x)$ has at least three roots in $[a, b]$.

★29. The coyote population in Nevada was the same at three consecutive times $t_1, t_2, t_3$. Assume that the population $N(t)$ is a differentiable function of time $t$ and is nonconstant on $[t_1, t_2]$ and on $[t_2, t_3]$. Establish by virtue of the mean value theorem the existence of two times $T$, $T^*$ in $(t_1, t_2)$ and $(t_2, t_3)$, respectively, for which the coyote population decreases.

★30. (a) Let $f$ be differentiable on $(a, b)$ [and continuous on $[a, b]$]. Suppose that, for all $x$ in the open interval $(a, b)$, the derivative $f'(x)$ belongs to a certain set $S$ of real numbers. Show that for any two distinct points $x_1$ and $x_2$ in $(a, b)$ [in $[a, b]$], the difference quotient

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

belongs to $S$ as well.

(b) Use (a) to prove consequence 1 of the mean value theorem.

(c) What does (a) tell you if $S = [a, b)$? If $S = (0, \infty)$?

## Review Exercises for Chapter 3

On what intervals are the functions in Exercises 1–4 continuous?

1. $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$
2. $f(x) = \dfrac{1}{\sqrt{x^3 - 1}}$
3. $f(x) = \dfrac{1}{x - 1} + \dfrac{1}{x - 2}$
4. $f(x) = \dfrac{x - 5}{x^2 - 3x + 2}$
5. Explain why the function $h$ given by

$$h(x) = \begin{cases} -x^3 + 5 & \text{for } x \leq 2 \\ x^2 - 7 & \text{for } x > 2 \end{cases}$$

is continuous at 2. Is $h$ continuous on the whole real line?

6. Let $f(x) = [1/x] + [(x^2 - 1)/x]$. Can you define $f(0)$ so that the resulting function is continuous at all $x$?
7. Find a function which is continuous on the whole real line and differentiable for all $x$ except 1, 2, and 3. (A sketch will do.)

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8. Sketch the graph of a function that has one vertical asymptote at $x = 1$, two maximum points, and no other critical points.
9. Show that $f(x) = x^4 + x^2 + x - 2$ has a root between $x = 0$ and $x = 1$.
10. Show that $f(x) = x^5 + x^3 + 2x + 1$ has a root between $x = -1$ and $x = 0$.
11. Find a solution to the equation in Exercise 9 to within two decimal places using the method of bisection.
12. Find a solution to the equation in Exercise 10 to within two decimal places using the method of bisection.

In Exercises 13–16, determine the intervals on which the given function is increasing and decreasing.

13. $8x^3 - 3x^2 + 2$.
14. $5x^3 + 2x^2 - 3x + 10$.
15. $\dfrac{x}{1 + (x-1)^2}$.
16. $\dfrac{x}{x^2 - 3x + 2}$, $x \neq 1, 2$.
17. Suppose that the British inflation rate (in percent per year) from 1990 to 2000 is given by the function $B(t) = 10(t^3/100 - t^2/20 - t/4 + 1)$, where $t$ is the time in years from January 1, 1993. When is the inflation rate increasing? How might a politician react to minimum points? Maximum points? Inflection points?
18. A paint can is kicked off the roof of a 156-foot building, its distance $S$ from the ground being given by $S = 156 + 4t - 16t^2$ ($S$ in feet, $t$ in seconds).
    (a) How high up does the can go before it falls downward?
    (b) At what time does the can return to roof level?
    (c) Find the velocity of the can when it collides with the ground.
19. The rate of growth of a tree between $t = 0$ and $t = 100$ years is given by the formula $r(t) = 10^{-6}(t^3 - 75t^2) + 10$ inches per year. When is the growth slowing down? speeding up?
20. A bicycle is moving at $(\frac{1}{16}t^3 + 4t^2 - 3t + 2)$ miles per hour, where $t$ is the time in hours, $0 \leq t \leq 2$. Does the bicycle change direction during this period?

In Exercises 21–24, find the critical points of the given function and determine if they are local maxima, minima, or neither.

21. $2x^3 - 5x^2 + 4x + 3$
22. $-8x^2 + 2x - 3$
23. $(x^2 + 2x - 3)^2$
24. $(x^3 - 12x + 1)^6$

For each of the functions in Exercises 25–30, answer the questions below and draw a graph:

(i) Where is $f$ continuous?
(ii) Where is $f$ differentiable?
(iii) On which intervals is $f$ increasing? Decreasing?
(iv) Where is $f$ concave upward or downward?
(v) What are the critical points, endpoints, local maximum and minimum points, and inflection points for $f$?

25. $f(x) = -7x^3 + 2x^2 + 15$ on $(-\infty, \infty)$.
26. $f(x) = 6x^2 + 3x + 4$ on $(-\infty, \infty)$.
27. $f(x) = x^3 - 2x + 1$ on $[-1, 2]$.
28. $f(x) = x^4 - 3x^3 + x^2$ on $[-6, 6]$.
29. $f(x) = \dfrac{3x}{2 - 5x}$ on its domain.
30. $f(x) = \dfrac{x}{1 + x^{2n}}$ on $(-\infty, \infty)$, $n$ is a positive integer.

Explain why each statement in Exercises 31–40 is true or false: (Justify if true, give a counterexample if false.)

31. Every continuous function is differentiable.
32. If $f$ is a continuous function on $[0, 2]$, and $f(1) = -1$ and $f(2) = 1$, then there must be a point $x$ in $[0, 1]$ where $f(x) = 0$.
33. If a function is increasing at $x = 1$ and at $x = 2$, it must be increasing at every point between 1 and 2.
34. If a differentiable function $f$ on $(-\infty, \infty)$ has a local maximum point at $x = 0$, then $f'(0) = 0$.
35. $f(x) = x^4 - x^3$ is increasing at zero.
36. $f(x) = x^3 + 3x$ takes its largest value on $[-1, 1]$ at an endpoint.
37. Parabolas never have inflection points.
38. All cubic functions $y = ax^3 + bx^2 + cx + d$, $a \neq 0$ have exactly one inflection point.
39. $f(x) = 3/(5x^2 + 1)$ on $(-\infty, \infty)$ has a local maximum at $x = 0$.
40. $y = x^5$ has an inflection point at $x = 0$.

Sketch the graphs of the functions in Exercises 41–50.

41. $x^3 + 3x + 2$
42. $x^5 - 3x^4$
43. $\dfrac{1}{x + 3}$
44. $\dfrac{1}{1 + x^4}$
45. $\dfrac{1 - x^3}{1 + 2x^3}$
46. $\dfrac{x - 5}{x^2 - 3x + 2}$
47. $x^{2/3} + (x - 1)^{2/3}$
48. $x^{4/3}$
49. $x(x - 1)^{3/2}$
50. $x(x - 1)^{2/3}$

In Exercises 51–56, find the maximum and minimum *value* of each function on the designated interval.

51. $f(x) = 3x^3 + x^2 - x + 5$; $[-2, 2]$
52. $f(x) = 2x^3 + 5x^2 - 4x$; $[-2, 2]$
53. $f(x) = \dfrac{5x^4 + 1}{x^2 + 1}$; $[-1, 1]$
54. $f(x) = \dfrac{x^3}{x^2 - 4}$; $[-1, 1]$
55. $f(x) = \dfrac{x^3}{x^2 - 4}$; $(-2, 2)$
56. $f(x) = -x^4 + 8x^2 + 2$; $(-\infty, \infty)$

In Exercises 57–62, sketch the graph of the indicated function on the designated interval.

57. The function in Exercise 51.
58. The function in Exercise 52.
59. The function in Exercise 53.
60. The function in Exercise 54.
61. The function in Exercise 55.
62. The function in Exercise 56.

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63. A wooden picture frame is to be 2 inches wide on top and bottom and 1 inch wide on the sides. Assuming the cost of a frame to be proportional to its front surface area, find the dimensions of the cheapest frame which will surround an area of 100 square inches.
64. At time $t$, a rectangle has sides given by $l(t) = 2 + t$ and $w(t) = 1 + t^2$, for $-1 \leqslant t \leqslant 1$. (a) When does this rectangle have minimum area? (b) When is the area shrinking the fastest?
65. Find the dimensions of the right circular cylinder of greatest volume that can be inscribed in a given right circular cone. Express your answer in terms of the height $h$ of the cone and the radius $r$ of the base of the cone.
66. A rectangular box with square bottom is to have a volume of 648 cubic centimeters. The top and bottom are to be padded with foam and pressboard, which costs three times as much per square centimeter as the fiberboard used for the sides. Which dimensions produce the box of least cost?
67. A box company paints its open-top square-bottom boxes white on the bottom and two sides and red on the remaining two sides. If red paint costs 50% more than white paint, what are the dimensions of the box with volume $V$ which costs least to paint? In what sense is the "shape" of this box independent of $V$?
68. Find the maximum area an isosceles triangle can have if each of its equal sides has a length of 10 centimeters.
69. The material for the top, bottom, and lateral surface of a tin can costs $\frac{1}{20}$ of a cent per square centimeter. The cost of sealing the top and bottom to the lateral surface is $p$ cents times the total length in centimeters of the rims (see Fig. 3.R.1) which are to be sealed. Find (in terms of $p$) the dimensions of the cheapest can which will hold a volume of $V$ cubic centimeters. Express your answer in terms of the solution of a cubic equation; do not solve it.

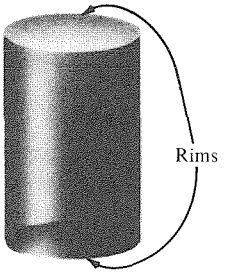


**Figure 3.R.1.** Minimize the cost of making the can.

70. We quote from V. Belevitch, *Classical Network Theory* (Holden-Day, 1968, p. 159): "When a generator of e.m.f. $e$ and internal *positive* resistance $R_i$ is connected to a positive load resistance $R$, the active power dissipated in the load

$$w = |e|^2 R/(R + R_i)^2, \qquad e \text{ a constant},$$

is maximum with respect to $R$ for $R = R_i$, that is, when the load resistance is *matched* to the internal resistance of the generator."

Verify this statement. (You do not need to know any electrical engineering.)

71. A fixed-frequency generator producting 6 volts is connected to a coil of 0.05 henry, in parallel with a resistor of 0.5 ohm and a second resistor of $x$ ohms. The power $P$ is given by

$$P = \frac{36x}{(\pi/10)^2 + (x + 1/2)^2}.$$

Find the maximum power and the value of $x$ which produces it.

72. The bloodstream drug concentration $C(t)$ in a certain patient's bloodstream $t$ hours after injection is given by

$$C(t) = \frac{16t}{(10t + 20)^2}.$$

Find the maximum concentration and the number of hours after injection at which it occurs.

73. In a drug-sensitivity problem, the change $T$ in body temperature for an $x$-milligram injection is given by

$$T(x) = x^2\left(1 - \frac{x}{4}\right), \qquad 0 \leqslant x \leqslant 4.$$

(a) Find the *sensitivity* $T'(x)$ when $x = 2$.
(b) Graph $T$ versus $x$.

74. The power $P$ developed by the engine of an aircraft flying at a constant (subsonic) speed $v$ is given by

$$P = \left(cv^2 + \frac{d}{v^2}\right)v,$$

where $c > 0$, $d > 0$.
(a) Find the speed $v_0$ which minimizes the power.
(b) Let $Q(t)$ denote the amount of fuel (in gallons) the aircraft has at time $t$. Assume the power is proportional to the rate of fuel consumption. At what speed will the flight time from takeoff to fuel exhaustion be maximized?

75. Prove that for any positive numbers $a, b, c$,

$$\frac{a + b + c}{3} \geqslant \sqrt[3]{abc}$$

by minimizing $f(x) = (a + b + x)/\sqrt[3]{abx}$.

76. Find the number $x$ which best approximates 1, 2, 3, and 5 in the sense that the sum of the *fourth* powers of the differences between $x$ and each number is minimized. Compute $x$ to within 0.1 by using the bisection method.
77. A manufacturer of hand calculators can produce up to 50,000 units with a wholesale price

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of \$16 for a fixed cost of \$9000 plus \$11.50 for each unit produced. Let $x$ stand for the number of units produced.

(a) Explain why the *total revenue* $R$, *total cost* $C$ and *profit* $P$ must satisfy the equations

$$R = 16x, \qquad C = 9000 + (11.5)x,$$

$$P = R - C = (4.5)x - 9000.$$

(b) The *break-even point* is the production level $x$ for which the profit is zero. Find it.

(c) Determine the production level $x$ which corresponds to a \$4500 profit.

(d) If more than 50,000 units are produced, the revenue is $R = 16x - \frac{4}{9} \times 10^{-6}x^2$, but the cost formula is unchanged. Find the $x$ that maximizes profits.

78. A manufacturer sells $x$ hole punchers per week. The weekly cost and revenue equations are

$$C(x) = 5000 + 2x,$$

$$R(x) = 10x - \frac{x^2}{1000}, \qquad 0 \leqslant x \leqslant 8000$$

(a) Find the minimum cost.

(b) Find the maximum revenue.

(c) Define the profit $P$ by the formula $P(x) = R(x) - C(x)$. Find the maximum profit.

79. A homeowner plans to construct a rectangular vegetable garden with a fence around it. The garden requires 800 square feet, and one edge is on the property line. Three sides of the fence will be chain-link costing \$2.00 per linear foot, while the property line side will be inexpensive screening costing \$.50 per linear foot. Which dimensions will cost the least?

80. A rental agency for compact cars rents 96 cars each day for \$16.00 per day. Each dollar increase in the rental rate results in four fewer cars being rented.

(a) How should the rate be adjusted to maximize the income?

(b) What is the maximum income?

81. The Smellter steel works and the Green Copper Corporation smelter are located about 40 miles apart. Particulate matter concentrations in parts per million theoretically decrease by an inverse square law, giving, for example,

$$C(x) = \frac{k}{x^2} + \frac{3k}{(40 - x)^2}$$

$$(1 \leqslant x \leqslant 39), \quad k \text{ a constant},$$

as the concentration $x$ miles from Smellter. This model assumes that Green emits three times more particulate matter than Smellter.

(a) Find $C'(x)$ all critical points of $C$.

(b) Assuming you wished to build a house between Smellter and Green at the point of least particulate concentration, how far would you be from Smellter?

82. Saveway checkers have to memorize the sale prices from the previous day's newspaper advertisement. A reasonable approximation for the percentage $P$ of the new prices memorized after $t$ hours of checking is $P(t) = 96t - 24t^2$, $0 \leqslant t \leqslant 3$.

What is the maximum percentage of the list memorized during the first 3 hours?

83. If $f'''(x) = 0$ on $(-\infty, \infty)$, show that there are constants $A$, $B$, and $C$ such that $f(x) = Ax^2 + Bx + C$.

84. Verify the horserace theorem for $f_1(x) = x^2 + x - 2$ and $f_2(x) = x^3 + 3x^2 - 2x - 2$ on $[0, 1]$.

85. Persons between 30 and 75 inches in height $h$ have average weight $W = \frac{1}{2}(h/10)^3$ lbs.

(a) What is the average weight of a person 5 feet 2 inches in height?

(b) A second grade child grows from 48 inches to 50 inches. Use the linear approximation to estimate his approximate weight gain and compare with a direct calculation.

86. A storage vessel for a chemical bleach mixture is manufactured by coating the inside of a thin hollow plastic cube with fiberglass. The cube has 12-inch sides, and the coating is $\frac{1}{5}$ inch thick. Use the volume formula $V = x^3$ and the mean value theorem to approximate the volume of the fiberglass coating. [*Hint*: The fiberglass volume is $V(12) - V(11.8)$.]

87. (a) Show that the quotient $f/g$ has a critical point when the ratios $f'/f$ and $g'/g$ are equal.

(b) Find a similar criterion for a product $fg$ to have a critical point.

88. (Refer to Review Problem 30 of Chapter 1 and Fig. 1.R.3.) Let $P$, with coordinates $(x_1, y_1)$, be inside the parabola $y = x^2$; that is, $y_1 > x_1^2$. Show that the path consisting of two straight lines joining $P$ to a point $(x, x^2)$ on the parabola and then $(x, x^2)$ to $(0, \frac{1}{4})$ has minimum length when the first segment is vertical.

89. The graph of a factored polynomial (such as $y = x(x-1)^2(x-2)^3(x-3)^7(x-4)^{12}$) near a root $r$ appears similar to that of the function $y = c(x - r)^n$, where $c$ and $n$ are chosen appropriately for the root $r$.

(a) Let $y = x(x+1)^2(x-2)^4$. For values of $x$ near 2, the factor $x(x+1)^2$ is approximately $2(2+1)^2 = 18$, so $y$ is approximately $18(x-2)^4$. Sketch the graph near $x = 2$.

(b) Argue that near the roots 0, 1, 3 the equation $y = 10x(x-1)^3(x-3)^2$ looks geometrically like $y = -90x$, $y = 40(x-1)^3$, $y = 240(x-3)^2$, respectively. Use this information to help sketch the graph from $x = 0$ to $x = 4$.

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★90. The *astroid* $x^{2/3} + y^{2/3} = 1$ is a planar curve which admits no self-intersections, but it has four cusp points.
   (a) Apply symmetry methods to graph the astroid.
   (b) Divide the astroid into two curves, each of which is the graph of a continuous function. Find equations for these functions.
   (c) Find the points where $dy/dx$ is not defined and compare with the cusp points. Use implicit differentiation.
   (d) Explain why no tangent line exists at the cusp points.

★91. A function $f$ is said to majorize a function $g$ on $[a, b]$ if $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$.
   (a) Show by means of a graph that $x$ majorizes $x^2$ on $[0, 1]$.
   (b) Argue that "$f$ majorizes $g$ on $[a, b]$" means that the curve $y = f(x)$, $a \leqslant x \leqslant b$, lies on or above the curve $y = g(x)$, $a \leqslant x \leqslant b$.
   (c) If $m > n > 0$, then $x^n$ majorizes $x^m$ on $[0, 1]$. Explain fully.
   (d) If $m > n > 0$, $(x - a)^n$ majorizes $(x - a)^m$ on $[a, a + 1]$. Why?
   (e) Given that $m \neq n$, $m > 0$, and $n > 0$, determine on which intervals $(x - a)^n$ majorizes $(x - a)^m$ (or conversely).
   (f) Graph the functions $y = x - 1$, $y = (x - 1)^2$, $y = (x - 1)^3$, $y = (x - 1)^4$, $y = (x - 1)^5$ on the same set of axes for $-1 \leqslant x \leqslant 3$.

★92. Let $f(x) = 1/(1 + x^2)$.
   (a) For which values of $c$ is the function $f(x) + cx$ increasing on the whole real line? Sketch the graph of $f(x) + cx$ for one such $c$.
   (b) For which values of $c$ is the function $f(x) - cx$ decreasing on the whole real line? Sketch the graph of $f(x) - cx$ for one such $c$.
   (c) How are your answers in parts (a) and (b) related to the inflection points of $f$?

★93. Let $f$ be a nonconstant polynomial such that $f(0) = f(1)$. Prove that $f$ has a strict local minimum or a strict local maximum point somewhere in the interval $(0, 1)$.

★94. Prove that, given any $n$ numbers $a_1, \ldots, a_n$, there is a uniquely determined number $x$ for which the sum of the fourth powers of the differences between $x$ and the $a_i$'s is minimized. [*Hint*: Use the second derivative.]

★95. Prove the following intermediate value theorem for derivatives: If $f$ is differentiable at all points of $[a, b]$, and if $f'(a)$ and $f'(b)$ have opposite signs, then there is a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$. [The example given in Review Exercise 84, Chapter 5, shows that this theorem does not follow from the intermediate value theorem of Section 3.1.]

★96. Let $f$ be differentiable function on $(0, \infty)$ such that all the tangent lines to the graph of $f$ pass through the origin. Prove that $f$ is linear. [*Hint*: Consider the function $f(x)/x$.]

★97. Let $f$ be continuous on $[3, 5]$ and differentiable on $(3, 5)$, and suppose that $f(3) = 6$ and $f(5) = 10$. Prove that, for some $x_0$ in the interval $(3, 5)$, the tangent line to the graph of $f$ at $x_0$ passes through the origin. [*Hint*: Consider the function $f(x)/x$.] Illustrate your result with a sketch.

★98. A function defined on $(a, b)$ is called *convex* when the following inequality holds for $x$, $y$ in $(a, b)$ and $t$ in $[0, 1]$:

$$f(tx + (1 - t)y) \leqslant tf(x) + (1 - t)f(y).$$

Demonstrate by the following graphical argument that if $f''$ is continuous and positive on $(a, b)$, then $f$ is convex:
   (a) Show that $f$ is concave upward at each point of $(a, b)$.
   (b) By drawing a graph, convince yourself that the straight line joining two points on the graph lies above the graph between those points.
   (c) Use the fact in (b) to deduce the desired inequality.

★99. Suppose that $f$ is continuous on $[a, b]$, $f(a) = f(b) = 0$, and $x^2 f''(x) + 4xf'(x) + 2f(x) \geqslant 0$ for $x$ in $(a, b)$. Prove that $f(x) \leqslant 0$ for $x$ in $[a, b]$.

★100. A rubber cube of incompressible material is pulled on all faces with a force $T$. The material stretches by a factor $v$ in two directions and contracts by a factor $v^{-2}$ in the other. By balancing forces, one can establish *Rivlin's equation*:

$$v^3 - \frac{T}{2\alpha} v^2 + 1 = 0,$$

where $\alpha$ is a constant (analogous to the spring constant for a spring). Show that Rivlin's equation has one (real) solution if $T < 3\sqrt[3]{2}\,\alpha$ and has three solutions if $T > 3\sqrt[3]{2}\,\alpha$.

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